

Varianta A

Vypočtete integrál $\int_{-2}^{-1} \frac{(x-3) dx}{6x^2 - x - 1}$.

ŘEŠENÍ: Integrovanou funkci rozložíme na parciální zlomky

$$\frac{x-3}{6x^2-x-1} = \frac{2}{3x+1} - \frac{1}{2x-1}.$$

Pak dostaneme

$$\begin{aligned} \int_{-2}^{-1} \frac{(x-3) dx}{6x^2-x-1} &= \int_{-2}^{-1} \left(\frac{2}{3x+1} - \frac{1}{2x-1} \right) dx = \\ &= \left[\frac{2}{3} \ln |3x+1| - \frac{1}{2} \ln |2x-1| \right]_{-2}^{-1} = \frac{2}{3} \ln 2 - \frac{1}{2} \ln 3 - \frac{1}{6} \ln 5. \end{aligned}$$

Najdětete řešení Cauchyovy úlohy

$$x_1' = x_1 + 2x_2, \quad x_2' = 3x_1 - 4x_2, \quad x_1(0) = 3, \quad x_2(0) = 5.$$

ŘEŠENÍ: Jedná se o homogenní soustavu dvou lineárních diferenciálních rovnic prvního řádu $\mathbf{x}' = \mathbf{A}\mathbf{x}$, kde $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$. Její charakteristická rovnice

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix} = \lambda^2 + 3\lambda - 10 = 0,$$

má řešení $\lambda_1 = 2$ a $\lambda_2 = -5$. Těmto vlastním číslům matice \mathbf{A} odpovídají vlastní vektory

$$\lambda_1 = 2 \longrightarrow \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \lambda_2 = -5 \longrightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Tedy soustava má dvě lineárně nezávislá řešení $\mathbf{x}_1 = e^{2t}\mathbf{v}_1$ a $\mathbf{x}_2 = e^{-5t}\mathbf{v}_2$. Obecné řešení uvedené soustavy proto je

$$\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2, \quad \text{tj.} \quad \begin{aligned} x_1 &= 2C_1e^{2t} + C_2e^{-5t}, \\ x_2 &= C_1e^{2t} - 3C_2e^{-5t}, \end{aligned}$$

kde C_1 a C_2 jsou konstanty. Pro ty získáme z počátečních podmínek rovnice

$$2C_1 + C_2 = 3, \quad C_1 - 3C_2 = 5, \quad \text{tj.} \quad C_1 = 2, \quad C_2 = -1.$$

Řešení Cauchyovy úlohy tedy je

$$x_1(t) = 4e^{2t} - e^{-5t}, \quad x_2(t) = 2e^{2t} + 3e^{-5t}.$$

Trojúhelník má základnu b a výška na tuto základnu je h . Vepište do něj obdélník, jehož jedna strana leží na základně a který má největší obsah.

ŘEŠENÍ: Označme x délku strany obdélníka, která leží na základně a v jeho výšku. Pak je jeho obsah $P = xv$. Z podobnosti trojúhelníků plyne vztah

$$\frac{h}{b} = \frac{h-v}{x}, \quad \text{tj.} \quad x = \frac{b(h-v)}{h}.$$

Když tento vztah dosadíme do P , dostaneme úlohu najít maximum funkce

$$P(v) = \frac{b(h-v)v}{h} \quad \text{na intervalu} \quad 0 \leq v \leq h.$$

Tato funkce může mít extrém v bodech, kde

$$P'(v) = \frac{b(h-2v)}{h} = 0, \quad \text{tj.} \quad v = \frac{1}{2}h$$

nebo v bodech $v = 0$ nebo $v = h$. Ale protože $P(0) = P(h) = 0$ a $P(\frac{1}{2}h) = \frac{1}{4}h^2$ je maximum v bodě $v = \frac{1}{2}h$ a $x = \frac{1}{2}b$.

Varianta B

Vypočtete integrál $\int_{-1}^0 \frac{(13 - 6x) dx}{3 - 7x - 6x^2}$.

ŘEŠENÍ: Integrovanou funkci rozložíme na parciální zlomky

$$\frac{13 - 6x}{2 - 7x - 6x^2} = \frac{3}{1 - 3x} + \frac{4}{2x + 3}$$

Pak dostaneme

$$\int_{-1}^0 \frac{(13 - 6x) dx}{3 - 7x - 6x^2} = \int_{-1}^0 \left(\frac{3}{1 - 3x} + \frac{4}{2x + 3} \right) dx = \left[-\ln |1 - 3x| + 2 \ln |2x + 3| \right]_{-1}^0 = 2 \ln 6.$$

Najdětete řešení Cauchyovy úlohy

$$x'' + 2x' + 5x = 0, \quad x(0) = 2, \quad x'(0) = 0.$$

ŘEŠENÍ: Jedná se o homogenní lineární diferenciální rovnici druhého řádu a konstantními koeficienty. Její charakteristická rovnice $\lambda^2 + 2\lambda + 5 = 0$ má řešení $\lambda_{1,2} = -1 \pm 2i$. Proto jsou dvě lineárně nezávislá řešení rovnice $x_1 = e^{-t} \cos 2t$ a $x_2 = e^{-t} \sin 2t$ a její obecné řešení

$$x(t) = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t,$$

kde C_1 a C_2 jsou konstanty. Z počátečních podmínek pro ně dostaneme rovnice

$$C_1 = 2, \quad -C_1 + 2C_2 = 0, \quad \text{tj.} \quad C_1 = 2, \quad C_2 = 1.$$

Tedy řešení Cauchyovy úlohy je

$$x(t) = e^{-t} (2 \cos 2t + \sin 2t).$$

Do koule s poloměrem R vepište válec, který má největší objemem.

ŘEŠENÍ: Označme r poloměr a v výšku hledaného válce. Pak je jeho objem roven $V = \pi r^2 v$. Dále musí být $r^2 + (\frac{1}{2}v)^2 = R^2$, neboli $r^2 = R^2 - \frac{1}{4}v^2$. Když tento vztah dosadíme do V dostaneme objem V válce jako funkci proměnné v , tj.

$$V(v) = \pi \left(R^2 - \frac{1}{4}v^2 \right) v.$$

Úloha tedy je najít maximum funkce $V(v)$ na intervalu $0 \leq v \leq 2R$. Tato funkce může mít maximum v bodě, kde je

$$V'(v) = \pi \left(R^2 - \frac{3}{4}v^2 \right) = 0, \quad \text{tj.} \quad v = \frac{2}{\sqrt{3}} R$$

nebo v bodech $v = 0$ nebo $v = 2R$. Protože $V(0) = V(2R) = 0$ a $V\left(\frac{2}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}} \pi R^3$, je maximum v bodě $v = \frac{2}{\sqrt{3}} R$ a $r = \sqrt{\frac{2}{3}} R$.

Test z matematiky – 31.1.2019

Vypočtete integrál $\int_{-3}^{-1} \frac{(x+6) dx}{4x^2 - x - 3}$.

ŘEŠENÍ: Integrovanou funkci rozložíme na parciální zlomky

$$\frac{x+6}{4x^2 - x - 3} = \frac{1}{x-1} - \frac{3}{4x+3}.$$

Pak dostaneme

$$\begin{aligned} \int_{-3}^{-1} \frac{(x+6) dx}{4x^2 - x - 3} &= \int_{-3}^{-1} \left(\frac{1}{x-1} - \frac{3}{4x+3} \right) dx = \\ &= \left[\ln|x-1| - \frac{3}{4} \ln|4x+3| \right]_{-3}^{-1} = \frac{3}{2} \ln 3 - \ln 2. \end{aligned}$$

Najdětete řešení Cauchyovy úlohy

$$x'' + 8x' + 17x = 0, \quad x(0) = 1, \quad x'(0) = -2.$$

ŘEŠENÍ: Jedná se o homogenní lineární diferenciální rovnici druhého řádu a konstantními koeficienty. Její charakteristická rovnice $\lambda^2 + 8\lambda + 17 = 0$ má řešení $\lambda_{1,2} = -4 \pm i$. Proto jsou dvě lineárně nezávislá řešení rovnice $x_1 = e^{-4t} \cos t$ a $x_2 = e^{-4t} \sin t$ a její obecné řešení

$$x(t) = C_1 e^{-4t} \cos t + C_2 e^{-4t} \sin t,$$

kde C_1 a C_2 jsou konstanty. Z počátečních podmínek pro ně dostaneme rovnice

$$C_1 = 1, \quad -4C_1 + C_2 = -2, \quad \text{tj.} \quad C_1 = 1, \quad C_2 = 2.$$

Tedy řešení Cauchyovy úlohy je

$$x(t) = e^{-4t} (\cos t + 2 \sin t).$$

Ze všech pravoúhlých trojúhelníků, které mají součet délek přepony a jedné odvěsny roven 3 m, určete ten, který má největší obsah.

ŘEŠENÍ: Označme a , b odvěsny a c přeponu hledaného trojúhelníka. Jeho obsah je $P = \frac{1}{2} ab$ a platí

$$a + c = 3, \quad a^2 + b^2 = c^2 = (3 - a)^2 = 9 - 6a + a^2.$$

Tedy $a = \frac{1}{6} (9 - b^2)$ a obsah trojúhelníka je

$$P(b) = \frac{(9 - b^2)b}{12}.$$

Tato funkce může nabývat největší hodnoty v bodě, kde je

$$P'(b) = \frac{1}{4} (3 - b^2) = 0, \quad \text{tj. pro } b = \sqrt{3}.$$

Extrémy by funkce $P(b)$ mohla mít ještě pro nejmenší a největší možné hodnoty b , tj. pro $b = 0$ a $b = 3$. Alepro tyto hodnoty b je $P(b) = 0$. Protože $P(\sqrt{3}) = \frac{\sqrt{3}}{2}$, nastává maximum pro $a = 1$, $b = \sqrt{3}$ a $c = 2$.

Test A

1. **Příklad:**

Na skladě je 100 výrobků. Kontrolou bylo zjištěno, že 12 výrobků má prošlou záruční lhůtu a 18 má poškozený obal. Dobrých výrobků bylo 75. Jaká je pravděpodobnost, že náhodně vybraný výrobek má obě jmenované vady?

[množin. diagram, $P = 0.05$]

POSTUP ŘEŠENÍ:

Množinový diagram pro záruku a obal, rovnice podle podmínek.

(Nebo: ... výrobek nemá prošlou záruku, jestliže má nepoškozený obal)?

[$P = 75/82$]

2. **Příklad:**

Zkonstruuje teoretické četnosti pro test rovnoměrnosti výskytu dopravních nehod, jestliže v určitém sledovaném období bylo zaznamenáno 56 nehod během všedních dnů, 5 nehod o sobotách a 18 nehod během nedělí. Vypočítejte hodnotu χ^2 statistiky pro tento test.

[$E = (5, 1, 1) \cdot \frac{79}{7}$, $\chi^2 = 7.4987$]

POSTUP ŘEŠENÍ:

Teoretické četnosti konstruuje tak, aby byl stejný počet pozorování (stejný součet) a aby platila rovnoměrnost - tj. hodnoty četností byly úměrné délkám intervalů.

3. **Příklad:**

Metodou momentů sestrojte statistiku pro odhad parametru p alternativního rozdělení s pravděpodobnostní funkcí

$$f(x) = p^x(1-p)^{1-x} \quad \text{pro } x \in \{0; 1\}.$$

[$\hat{p} = \bar{x}$]

POSTUP ŘEŠENÍ:

Porovnat první moment souboru (spočtený podle definice z hustoty) a výběru (výběrový průměr).

Test B

1. Příklad:

Dva soupeři hází střídavě kostkou. Vyhrává ten, kdo první hodí šestku. Jaká je pravděpodobnost, že vyhraje ten, kdo začínal házet?

$$[\text{geom. řada } a_1 = 1/6 \text{ } q = 25/36, P=6/11]$$

POSTUP ŘEŠENÍ:

První hod: výhra $1/6$; druhý hod: nic; třetí hod: výhra $(5/6)(6/6)(1/6)$ - před nesměly padnout šestka, atd
→ geometrická řada $a_1 = 1/6, q = (5/6)^2$.

2. Příklad:

Zkonstruuje teoretické četnosti pro test nezávislosti počtu dopravních nehod (N: nula, 1-10, nad 10) a pohlaví řidiče (P: muž, žena) pro data z kontingenční tabulky

$P \setminus N$	nula	1-10	nad 10
muž	26	5	15
žena	34	12	3

$$[E = [29.05 \ 8.23 \ 8.72; 30.95 \ 8.77 \ 9.28]]$$

POSTUP ŘEŠENÍ:

Spočteme: relativní četnosti sdružené, marginály, součin marginál jako sdruženou pro nezávislé veličiny, přepočteme na absolutní četnosti. Součet prvků je 96; marginály (0.63 0.18 0.19) a (0.48 0.52); sdružená pro nezávislé [0.31 0.087 0.09; 0.33 0.09 0.10].

3. Příklad:

Dokažte nestrannost výběrového průměru jako statistiky pro odhad střední hodnoty souboru.

$$[E[\frac{1}{n} \sum X_i] = \frac{1}{N} \sum E[X_i] = \mu]$$

POSTUP ŘEŠENÍ:

Viz výsledek.

DATE: January 31, 2019

NAME:

TOTAL: _____ /15 pts

PART 1 (5 pts, 1 each sentence)

Each of these sentences contains a mistake. Find the mistake and correct it:

1. My cousin Jamie is a really **annoyed** **ANNOYING** person when he talks about his job, but I'm always interested in listening to his stories about his trips around the world.
2. It was such **A** long time since Dawn had seen her grandfather that there was a lot for them to talk about.
3. The person I most admire in my family is my mother because she's **so** **SUCH A** supportive person and always had enough time to help me.
4. When the twins were **too** **VERY** young, we never had enough money to buy them many things.
5. At the age of ten, David had an amazing singing voice, but he usually felt very **embarrassing** **EMBARRASSED** when he sang in front of other people.

PART 2 (5 pts, 1 each sentence)

Complete the second sentence so that it has a similar meaning to the first sentence, using the word given. Do not change the word given. You must use between two and five words, including the word given.

1. In the UK, there are fewer Italian restaurants than Indian ones.

AS

In the UK, there aren't **AS MANY ITALIAN RESTAURANTS AS** Indian ones.

2. Generally speaking, we don't eat as much meat as we used to.

LESS

Generally speaking, we **MUCH LESS MEAT THAN** we used to.

3. The new oven isn't as efficient as the old one.

WORKED

Our old oven **WORKED BETTER THAN** the new one.

4. I hate washing-up more than any other job in the kitchen.

LEAST

Washing-up is my **LEAST FAVOURITE JOB** in the kitchen.

5. It's easier to cook pasta than most other dishes.

ONE

Pasta is **ONE OF THE EASIEST DISHES** to cook.

PART 3 (5 pts, 0.5 each gap)

Use the word given in **CAPITALS** at the end of some of the lines to form a word that fits in the gap in the same line.

<p>It was my 18th birthday, so my dad decided to take the family out for what we hoped would be a (0) memorable meal. He had chosen a (1) LUXURIOUS restaurant in the town centre where you sit beside a pool in really (2) COMFORTABLE chairs surrounded by exotic plants. In my family we are quite (3) ADVENTUROUS when it comes to eating and enjoying trying out new dishes.</p> <p>My preference is for really (4) SPICY food.</p> <p>When we saw the size of the menu we were really surprised – it was probably the (5) LONGEST menu I had ever seen and it had so many sections that we found it quite (6) CONFUSING.</p> <p>Anyway, the waiter was very (7) HELPFUL and in the end we each made our (8) CHOICE. I asked for a goat's cheese tart which I didn't think looked very (9) ATTRACTIVE when I saw it. Anyway, it tasted fantastic. All in all, it was a (10) WONDERFUL way to spend my birthday.</p>	<p>MEMORY LUXURY COMFORT ADVENTURE SPICE LONG CONFUSE HELP CHOOSE ATTRACT WONDER</p>
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DATE: June 12 2019

NAME:

TOTAL: /15 pts

PART 1 (7.5 pts, 1 each sentence used/0.5 not used)

You are going to read the introduction to a guidebook about the Yosemite National Park in the USA. Seven sentences have been removed from the article. For the gaps 21-27 choose the sentence A, B, C or D which fits best. Sentences appear in the options more than once. Each correct option can be used just ONCE. Among the options offered, there is ONE sentence that is NOT used.

Yosemite National Park

What exactly is Yosemite? Is it Bridaveil Fall thundering and pouring in early June? Is it a long summer's day at Tuolumne Meadows? Is it the ice-carved, rocky world of the high Sierra seen from Glacier Point? Obviously, Yosemite is all of these things and much, much more. It is an energetic walk over the Four Mile Trail. It is the smell of pine trees at Hogdon Meadow campground. It might also be a walk among some of the largest trees in the world. Our list could go on and on.

_____ 21 A _____ Roaring waterfalls, falling hundreds of feet, fascinate even the most bored traveller. Shining walls of towering rock face challenge the skills of hundreds of mountain climbers and capture the eyes and minds of thousands of visitors. Yosemite's rushing mountain streams, alpine landscape, forests and all the rest of its natural features combine to make this national park unique in the opinion of nearly every observer.

_____ 22 A _____ These earlier inhabitants of the region left traces of a lifestyle which depended upon the use of local plants and animals. Remains of that culture, on display in museums and books, sometimes seen in the surfaces of rock, recalled mainly in names upon the land, show us people's lives which were directly connected to this region.

_____ 23 B _____ From the earliest Spanish explorers who gave names to the general region, to the fur trappers, miners and others who came seeking paths through the Sierra Nevada or hoping for personal gain, Yosemite displays an exciting past which helps us understand the present. It is a story filled with characters who were impressed enough to stay, advertise, exploit and preserve.

_____ 24 C _____ Its geologic features are the product of time's hidden forces, carved out by glaciers and streams. Its birds and bears delight suburban America. Its buzzing mosquitoes remind us that we are not in a shopping mall. Its flower-filled meadows and tall trees remind us of the sheep and loggers who once looked out upon this scene.

_____ 25 B _____ While preserved for all to enjoy, perhaps not everyone can enjoy it at once. Occasionally crowded conditions disturb many first-time visitors. Yosemite Valley does not seem like the quiet place generally shown in photographs. An ever-increasing, demanding public raises the question – can any national park be all things to all people?

_____ 26 C _____ Bicycling in Yosemite alley, walking the John Muir Trail, skiing at Badger Pass or sitting quietly beside the Merced River are all possibilities. One can walk with freedom in the park, allowing closer examination of the natural surroundings. Alternatively, visitors to Yosemite can take shuttle buses and disembark for short adventures beyond the roadway or can go into informational museums.

Yosemite is a spectacular Sierra Nevada park. Yosemite is history, geology, Indians, scenery beyond compare, and conservation. Yosemite is part of America that we always want to experience and never want to lose. It has become a part of our imagination. We search in Yosemite for what we have not been able to find elsewhere.

_____ 27 A And that may explain why Yosemite is so popular

- 21 A Yosemite contains natural features which cannot fail to attract human attention
B Yosemite is well-known not just to Americans, but to people all over the world.
C Yosemite also shows us how the original native American people lived.
D Yosemite recalls a history, rich with colourful personalities and filled with dramatic events.
- 22 A Yosemite also shows us how the original native American people lived.
B Yosemite contains natural features which cannot fail to attract human attention.
C For today's visitors, Yosemite offers a source of pleasure and a choice of activity.
D Because of that, Yosemite is more than a park, it is an ideal.
- 23 A Yosemite might also be an example of a national park that is too successful, that has become too popular.
B Yosemite recalls a history, rich with colourful personalities and filled with dramatic events.
C Yosemite also shows us how the original native American people lived.
D Yosemite is well-known not just to Americans, but to people all over the world.
- 24 A Yosemite also shows us how how the original native American people lived.
B For today's visitors, Yosemite offers a source of pleasure and a choice of activity.
C Yosemite is also an example of wild America, in contrast to the America outside its boundaries.
D Yosemite might also be an example of a national park that is too successful, that has become too popular.
- 25 A Yosemite is also an example of wild America, in contrast to the America outside its boundaries.
B Yosemite might also be an example of a national park that is too successful, that has become too popular.
C Yosemite contains natural features which cannot fail to attract human attention.
D Yosemite recalls a history, rich with colourful personalities and filled with dramatic events.
- 26 A Yosemite contains natural features which cannot fail to attract human attention.
B Because of that, Yosemite is more than a park, it is an ideal.
C For today's visitors, Yosemite offers a source of pleasure and a choice of activity.
D Yosemite is well-known not just to Americans, but to people all over the world.
- 27 A Because of that, Yosemite is more than a park, it is an ideal.
B For today's visitors, Yosemite offers a source of pleasure and a choice of activity.
C Yosemite is well-known not just to Americans, but to people all over the world.
D Yosemite is also an example of wild America, in contrast to the America outside its boundaries.

PART 2 (7.5 pts, 0.5 each question)

Choose the answer which best fits the sentence.

28. D goes the bus; now we will have to walk!

- A. On time B. At once C. Early D. There

29. 'My daughter Mary tries to A to see me at least once a week,' Mrs Jones told the social worker.

- A. drop in B. call up C. come on D. call on

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30. D it was raining, we all went for a walk.
A. While B. In spite of C. Despite D. Although
31. I saw a thief take Joseph's wallet so I ran C him, but I didn't catch him.
A. into B. forward C. after D. for
32. Driving in London is supposed to be confusing but I didn't find it at B difficult.
A. first B. all C. once D. least
33. The police officer asked me why A so fast.
A. I had been driving B. was I driving C. I have been driving D. have I been driving
34. This is the first time I A Prague.
A. have visited B. am visiting C. visited D. visit
35. All students leaving college at the end of the term must leave their addresses so that their letters can be sent A.
A. forward B. through C. to D. on
36. When she was crossing the room the night-nurse happened to notice the old lady D to get out of bed.
A. has tried B. to try C. tried D. trying
37. You C go to the dentist's before your toothache gets worse.
A. rather B. ought C. better D. ought to
38. Her father won't C her drive his car.
A. allow B. leave C. let D. permit
39. There was nothing they could do B leave the car at the roadside where it had broken down.
A. unless B. but C. instead of D. than
40. If it hadn't been for Julia's intervention, we A lost.
A. would have got B. would had got C. would got D. would get
41. There is D to be frightened of the dog; he's quite harmless.
A. no point B. no fear C. any reason D. no need
42. We went to see the play last night, and D for Tony, we all enjoyed it very much.
A. apart B. aside C. besides D. except