1 Control with regression model

1.1 Derivation in pdf

Criterion

Optimal control needs criterion. We will use summation one

$$J = \sum_{t=1}^{N} J_t$$

where J_t is a penalization for time t. Mostly it is $J_t = y_t^2 + \omega u_t^2$.

We want to set $u_t,\ t = 1, 2, \cdots, N$ that minimizes J. But, J is a random variable, due to the output y_t. As random variable can take many different values it is not possible to speak about its minimization. So, we must minimize its estimate (which is expectation). So the minimized criterion is

$$E\left[J|d(0)\right] = E\left[\sum_{t=1}^{N} J_t|d(0)\right]$$

where in condition of the expectation is our preliminary knowledge - prior data.

Remark

For $N = 1$ we obtain one-step control. Here, we optimize control only for the next output. This control is dangerous, because the controller does not take into account future evolution of the system and to act best in one step it can generate too beg output. This can excite the system so much that it is not possible even to stabilize it in the future and the control fails.

Minimization

$$\min_{u_{1:N}} E\left[\varphi_{N+1}^* + \sum_{t=1}^{N} J_t|d(0)\right] =$$

$$= \min_{u_{1:(N-1)}} E\left[\min_{u_N} \underbrace{E\left[\varphi_{N+1}^* + J_N|u_N, d(N-1)\right]}_{\varphi_N^*} + \sum_{t=1}^{N-1} J_t \middle| d(0)\right] =$$

$$= \min_{u_{1:(N-1)}} E\left[\min_{u_N} \varphi_N + \sum_{t=1}^{N-1} J_t|d(0)\right] = \min_{u_{1:N}} E\left[\varphi_N^* + \sum_{t=1}^{N-1} J_t|d(0)\right]$$

which reproduces the initial form, only with $N \to N-1$ and where (due to the reproduction in general form)

Bellman equations

$$\varphi_t = E\left[\varphi_{t+1}^* + J_t | u_t, d(t-1)\right] \quad \text{expectation}$$
$$\varphi_t^* = \min_{u_t} \varphi_t \quad \text{minimization}$$

for $t = N,\ N-1,\ N-2,\ \cdots,\ 1$. Each minimization gives the formula for optimal control - it is $u_t = \arg\min \varphi_t(d(t-1))$. However, ti cannot be used immediately, because the data $d(t-1)$ is not known, yet. Only at time $t = 1$ we need data $d(0)$ and the control can start to be generated.

Comments

1. The operator form of expectation is brief but not explicit. We will show its integral form:

$$\min_{u_{1:N}} E\left[\varphi_{N+1}^* + \sum_{t=1}^{N} J_t | d(0)\right] =$$
$$= \min_{u_{1:N}} \int \cdots \int \left(\varphi_{N+1}^* + \sum_{t=1}^{N} J_t\right) f(y(N), u(N) | d(0))\, dy(N)\, du(N) =$$
$$= \min_{u_{1:N}} \int \cdots \int \int \int \left(\left[\varphi_{N+1}^* + J_N\right] + \sum_{t=1}^{N-1} J_t\right) f(y_N | u_N, d(N-1))\, f(u_N | d(N-1)) \times$$
$$\times f(y(N-1), u(N-1) | d(0))\, dy(N)\, du(N) =$$
$$= \min_{u_{1:(N-1)}} \left\{ \int \cdots \int \min_{u_N} \int \underbrace{\int \left(\varphi_{N+1}^* + J_t\right) f(y_N | u_N, d(N-1))\, dy_N}_{\varphi_N(u_N, d(N-1))} f(u_N | d(N-1))\, du_N + \right.$$
$$\left. \sum_{t=1}^{N-1} J_t f(y(N-1), u(N-1) | d(0))\, dy(N-1)\, du(N-1) \right\}$$

Minimum over u_N

$$\min_{u_N} \int \underbrace{\int \left(\varphi_{N+1}^* + J_t\right) f(y_N | u_N, d(N-1))\, dy_N}_{\varphi_N(u_N, d(N-1))} f(u_N | d(N-1))\, du_N =$$
$$= \min_{u_N} \int \varphi_N(u_N, d(N-1))\, f(u_N | d(N-1))\, du_N$$

$\rightarrow u_N^* = \arg\min_{u_N} \varphi_N$ and $f(u_N | d(N-1)) = \delta(u_N, u_N^*)$ - all u_t is concentrated into one point u_N^*.

1.2 Derivation for regression model

Regression model can be converted to state-space form (see lecture 2 - Regression model in state-space form).

$$x_t = Mx_{t-1} + Nu_t + w_t$$

where $x_t = [y_t, u_t, y_{t-1}, u_{t-1}, \cdots y_{t-n+1}, u_{t-n+1}]'$.

The penalty can be written as

$$y_t^2 + \omega u_t^2 = x_t' \Omega x_t \tag{1.1}$$

where Ω is a diagonal matrix

$$\Omega = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & 0 & & \\ & & & \cdots & \\ & & & & 0 \end{bmatrix}$$

Now the model and criterion is used in general Bellman equations, where we guess the form of $\varphi_{t+1}^* = x_t^{'} R_{t+1} x_t$

$$E\left[x_t^{'} R_{t+1} x_t + x_t^{'} \Omega x_t | u_t, d\left(t-1\right)\right] = E\left[x_t^{'} U x_t\right] =$$

$$= \left(Mx_{t-1} + Nu_t\right)' U \left(Mx_{t-1} + Nu_t\right) + \rho =$$

$$= x_{t-1}^{'} \underbrace{M'UM}_{C} x_{t-1} + 2u_t^{'} \underbrace{N'UM}_{B} x_{t-1} + u_t^{'} \underbrace{N'UN}_{A} u_t + \rho =$$

$$= u_t^{'} A u_t + 2u_t^{'} A \underbrace{A^{-1} B}_{S_t} x_{t-1} + x_{t-1}^{'} S_t^{'} A S_t x_{t-1} +$$

$$+ \underbrace{x_{t-1}^{'} C x_{t-1} - x_{t-1}^{'} S_t^{'} A S_t x_{t-1}}_{x_{t-1} R_t x_{t-1}} + \rho =$$

$$= \left(u_t + S_t x_{t-1}\right)' A \left(u_t + S_t x_{t-1}\right) + x_{t-1}^{'} R_t x_{t-1} + \rho$$

Optimal $u_t = S_t x_{t-1}$.

Recursion

$R_{N+1} = 0$

for $t = N,\, N-1,\, \cdots,\, 1$

$U = R_{t+1} + \Omega$
$A = N'UN$
$B = N'UM$
$C = M'UM$
$S_t = A^{-1}B$
$R_t = C - S_t' A S_t$
$u_t = S_t x_{t-1}.$

end

Remark

The penalty function (1.1) can be very easily extended to the following form

$$(y_t - s_t)^2 + \omega u_t^2 + \lambda (u_t - u_{t-1})^2$$

where the first term leads to the following the output y_t the prescribed set-point s_t and the last term introduces penalization of increments of the control variable. Penalizing the control increments calms control behavior and at the same time it does not result to steady-state deviation of the output and the set-point as it is when penalizing the whole control variable.

The solution how to introduce the above requirements for the control lies in construction of the penalization matrix as follows

$$\Omega = \begin{bmatrix} 1 & & & & & & -1 \\ & \omega + \lambda & & -\lambda & & & \\ & & 0 & & & & \\ & -\lambda & & \lambda & & & \\ & & & & \cdots & & \\ & & & & & 0 & \\ -1 & & & & & & 1 \end{bmatrix}$$

which is evident if we take into account that the criterion is

$$x_t' \Omega x_t$$

and $x_t = [y_t, u_t, y_{t-1}, u_{t-1}, \cdots, 1]$.