Dodatky ke slajdům

Regresní model ve stavovém tvaru

Ukážeme pro regresní model 2. řádu

$$y_t = b_0 u_t + a_1 y_{t-1} + b_1 u_{t-1} + a_2 y_{t-2} + b_2 u_{t-2} + k + e_t$$

Stavový tvar je

$$\begin{bmatrix} y_t \\ u_t \\ y_{t-1} \\ u_{t-1} \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & a_2 & b_2 & k \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ u_{t-1} \\ y_{t-2} \\ u_{t-2} \\ 1 \end{bmatrix} + \begin{bmatrix} b_0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_t + \begin{bmatrix} e_t \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Generování z kategorického rozdělení

Obecně platí

$$F(X) = u \;\to\; X = F^{-1}(u)$$

Vzorec

```
y = sum(cumsum(f)<rand(1,1,'u'))+1;
```

Vysvětlení: Máme pf

$$f(y) = [0.1,\ 0.6,\ 0.3]$$

Distribuční funkce je dána jako kumulativní součet (ve Scilabu `cumsum()`)

$$F(y) = [0.1,\ 0.7,\ 1]$$

Generujeme rovnoměrné `u=rand(1,1,'u')`a hledáme, kde platí

$$\underbrace{u < [0.1,\ 0.7,\ 1]}_{q}.$$

Příklad: pro $u = 0.653$ dostaneme

$$q = [1,\ 0,\ 0]$$

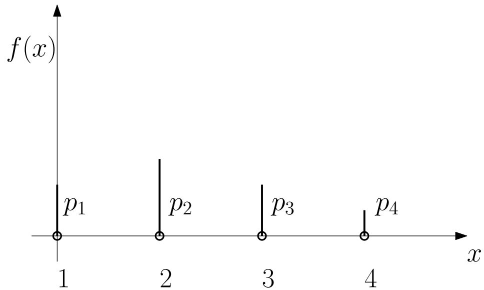
a

$$\sum [1,\ 0,\ 0] + 1 = 2$$

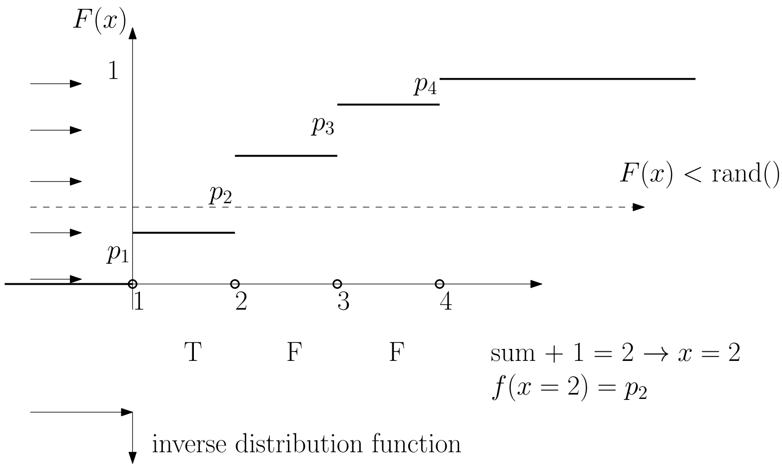
protože u padne do druhého intervalu.

Tak dostáváme hodnoty 0,1,2. Přidáme $+1$ a máme 1,2,3.

Probability function



Distribution function



Likelihood

Model: $f\left(y_t|\Theta\right)$

Data: $D = y_1, y_2, \cdots, y_N$

Když máme data chceme znát parametry

$$f\left(\Theta|D\right) \propto f\left(D|\Theta\right) = f\left(y_1, y_2, \cdots, y_N|\Theta\right) =$$

$$= \prod_{i=1}^{N} f\left(y_t|\Theta\right) = \underbrace{L_N\left(\Theta\right)}_{\text{Likelihood}} \quad \text{(nezávislost)}$$

Odhad exponenciálního modelu

Model

$$f\left(y|a\right)=a\exp\left(-ay\right)$$

součin modelů

$$f\left(y_1|a\right)f\left(y_2|a\right)=a\exp\left(-ay_1\right)a\exp\left(-ay_2\right)=$$

$$=a^2\exp\left(-a\left(y_1+y_2\right)\right)$$

odtud

$$\prod_{i=1}^{n}f\left(y_i|a\right)=a^n\exp\left(-a\sum_{i=1}^{n}y_i\right)$$

kde $S_n=\sum_{i=1}^{n}y_i$ a $\kappa_t=n$

Aposteriorní

$$f\left(a|y\left(t\right)\right)=a^{\kappa_t}\exp\left(-aS_t\right)$$

s přepočtem

$$S_t=S_{t-1}+y_t$$
$$\kappa_t=\kappa_{t-1}+1$$

se startem $S_0=0$, $\kappa_0=0$.

Bayes $f\left(a|y\left(t\right)\right)\propto f\left(y_t|a\right)f\left(a|y\left(t-1\right)\right)$

$$a^t\exp\left(-a\sum_{i=1}^{t}y_i\right)=a\exp\left(-ay_t\right)\,a^{t-1}\exp\left(-a\sum_{i=1}^{t-1}y_i\right)$$

a formálně

$$a^{\kappa_t}\exp\left(-aS_t\right)=a\exp\left(-ay_t\right)\,a^{\kappa_{t-1}}\exp\left(-aS_{t-1}\right)$$

Bodový odhad (ML)

$$\frac{d}{da} a^{\kappa} \exp(-aS) = \kappa a^{\kappa-1} \exp(-aS) - a^{\kappa} S \exp(-aS) = 0$$

$$\kappa - aS = 0 \quad \rightarrow \quad a = \frac{\kappa}{S} = \frac{1}{\bar{y}}$$

Apriorno

zvolíme

$$f\left(a|y\left(0\right)\right)=a^{\kappa_0}\exp\left(-aS_0\right)$$

kde S_0 je součet apriorních dat a κ_0 je jejich počet.

Máme-li informaci, že asi $a\doteq 5$, bude $y=\frac{1}{5}$. Nejdříve volíme sílu apriorna (třeba) $\kappa_0=20$ a potom $S_0=\sum y=\kappa_0\frac{1}{5}=4$.

Potom $\frac{1}{a}=\frac{S_0}{\kappa_0}=\frac{4}{20}=\frac{1}{5}$ a $a=5$ (tedy odhad z apriorna odpovídá naší informaci).

Podobně je to u celé exponenciální rodiny distribucí - např.

Binomické rozdělení

model

$$f\left(y|p\right) = \binom{N}{y} p^{y} \left(1-p\right)^{N-y}$$

součin dvou

$$\binom{N}{y_1} p^{y_1} \left(1-p\right)^{N-y_1} \binom{N}{y_2} p^{y_2} \left(1-p\right)^{N-y_2} =$$

$$\binom{N}{y_1}\binom{N}{y_2} p^{y_1+y_2} \left(1-p\right)^{2N-(y_1+y_2)}$$

obecný součin (likelihood)

$$\prod_{i=1}^{t} \binom{N}{y_i} p^{\sum_{i=1}^{t} y_i} \left(1-p\right)^{tN-\sum_{i=1}^{t} y_i} \propto p^{S_t} \left(1-p\right)^{\kappa_t N - S_t}$$

odhad

$$\hat{p}_t = \frac{S_t}{N\kappa_t}$$

Odhad nelineárního stavu - příklad

Model

$$x_{1;t} = \underbrace{\exp\{-x_{1;t-1} - x_{2;t-1}\} + u_t}_{g_1} + w_t$$

$$x_{2;t} = \underbrace{x_{1;t-1} - 0.3u_t}_{g_2} + w_{2;t}$$

$$y_t = x_{2;t} + v_t$$

Linearizace

$$g_1 = \exp\{-x_1 - x_2\} + u_t$$

$$g_2 = x_1 - 0.3u_t$$

$$g_1^{'} = \left[\frac{\partial g_1}{\partial x_1}, \frac{\partial g_1}{\partial x_2}\right] = \left[-\exp\{-x_1 - x_2\}, -\exp\{-x_1 - x_2\}\right]$$

$$g_2^{'} = \left[\frac{\partial g_2}{\partial x_1}, \frac{\partial g_2}{\partial x_2}\right] = [1, 0]$$

Platí

$$\bar{M} = g' = \begin{bmatrix} g_1^{'} \\ g_2^{'} \end{bmatrix} = \begin{bmatrix} -\exp\left\{-x_1 - x_2\right\}, & -\exp\left\{-x_1 - x_2\right\} \\ 1 & 0 \end{bmatrix}$$

$$F = g - \bar{M}\hat{x}_{t-1} = \begin{bmatrix} \exp\left\{-x_1 - x_2\right\} + u_t \\ x_1 - 0.3u_t \end{bmatrix} - \tilde{M}\hat{x}_{t-1}$$

Výsledný model je

$$x_t = \bar{M}x_{t-1} + F + w_t$$

$$y_t = \bar{A}x_t + v_t$$

kde $\bar{A} = [0,\ 1]$

Program

$$[xt, Rx, yp] = \text{Kalman}(xt, yt, ut, \bar{M}, 0, F, \bar{A}, 0, 0, Rw, Rv, Rx)$$

Řízení s regresním modelem

Regresní model převedeme na stavový

$$x_t = Mx_{t-1} + Nu_t + w_t$$

se stavem $x_t = [y_t, u_t, y_{t-1}, u_{t-1}, \cdots y_{t-n+1}, u_{t-n+1}]'$.

Penalizace

$$J_t = y_t^2 + \omega u_t^2 = x_t' \Omega x_t \tag{1}$$

Ω je

$$\Omega = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & 0 & & \\ & & & \cdots & \\ & & & & 0 \end{bmatrix}$$

Podle Bellmanových rovnic $\varphi^*_{t+1} = x^{'}_t R_{t+1} x_t$

$$E\left[x^{'}_t R_{t+1} x_t + x^{'}_t \Omega x_t | u_t, d\left(t-1\right)\right] = E\left[x^{'}_t U x_t\right] =$$

kde $U = R_{t+1} + \Omega$

$$= \left(M x_{t-1} + N u_t\right)' U \left(M x_{t-1} + N u_t\right) + \rho =$$

dosazen model

$$= x^{'}_{t-1} \underbrace{M'UM}_{C} x_{t-1} + 2u^{'}_t \underbrace{N'UM}_{B} x_{t-1} + u^{'}_t \underbrace{N'UN}_{A} u_t + \rho =$$

doplníme na čtverec

$$= u^{'}_t A u_t + 2u^{'}_t A \underbrace{A^{-1}B}_{S_t} x_{t-1} + x^{'}_{t-1} S^{'}_t A S_t x_{t-1} +$$

$$+\underbrace{x_{t-1}^{'}Cx_{t-1}-x_{t-1}^{'}S_t^{'}AS_tx_{t-1}}_{x_{t-1}R_tx_{t-1}}+\rho=$$

$$=\left(u_t+S_tx_{t-1}\right)'A\left(u_t+S_tx_{t-1}\right)+x_{t-1}^{'}R_tx_{t-1}+\rho$$

a minimalizujeme

Optimální řízení $u_t=-S_tx_{t-1}$.

Zobecnění

$$\left(y_t - s_t\right)^2 + \omega u_t^2 + \lambda \left(u_t - u_{t-1}\right)^2$$

$$\Omega = \begin{bmatrix} 1 & & & & & & -1 \\ & \omega + \lambda & & -\lambda & & & \\ & & 0 & & & & \\ & -\lambda & & \lambda & & & \\ & & & & \cdots & & \\ & & & & & 0 & \\ -1 & & & & & & 1 \end{bmatrix}$$

$$x_t^{'} \Omega x_t$$

se stavem $x_t = \left[y_t,\, u_t,\, y_{t-1},\, u_{t-1},\, \cdots, 1\right].$

Konstrukce Ω

$$[y_t, \quad u_t, \quad y_{t-1}, \; u_{t-1}, \; y_{t-2}, u_{t-2}, \; 1]$$

$$\begin{bmatrix} y_t \\ u_t \\ y_{t-1} \\ u_{t-1} \\ y_{t-2} \\ u_{t-2} \\ 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & & -1 \\ & \omega + \lambda & & -\lambda & & & \\ & & 0 & & & & \\ & -\lambda & & \lambda & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ -1 & & & & & & 1 \end{bmatrix}$$

Penalizace (podrobně)

$$[y_t, u_t, y_{t-1}, u_{t-1}, 1] \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} & \Omega_{24} & \Omega_{25} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & \Omega_{34} & \Omega_{35} \\ \Omega_{41} & \Omega_{42} & \Omega_{43} & \Omega_{44} & \Omega_{45} \\ \Omega_{51} & \Omega_{52} & \Omega_{53} & \Omega_{54} & \Omega_{55} \end{bmatrix} \begin{bmatrix} y_t \\ u_t \\ y_{t-1} \\ u_{t-1} \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \Omega_{11} y_t, & \Omega_{12} y_t, & \Omega_{13} y_t, & \Omega_{14} y_t, & \Omega_{15} y_t \\ + & + & + & + & + \\ \Omega_{21} u_t, & \Omega_{22} u_t, & \Omega_{23} u_t, & \Omega_{24} u_t, & \Omega_{25} u_t \\ + & + & + & + & + \\ \Omega_{31} y_{t-1}, & \Omega_{32} y_{t-1}, & \Omega_{33} y_{t-1}, & \Omega_{34} y_{t-1}, & \Omega_{35} y_{t-1} \\ + & + & + & + & + \\ \Omega_{41} u_{t-1}, & \Omega_{42} u_{t-1}, & \Omega_{43} u_{t-1}, & \Omega_{44} u_{t-1}, & \Omega_{45} u_{t-1} \\ + & + & + & + & + \\ \Omega_{51}, & \Omega_{52}, & \Omega_{53}, & \Omega_{54}, & \Omega_{55} \end{bmatrix} \begin{bmatrix} y_t \\ \\ u_t \\ \\ y_{t-1} \\ \\ u_{t-1} \\ \\ 1 \end{bmatrix} =$$

$$
= \begin{bmatrix}
\Omega_{11} y_t^2 + & \Omega_{12} y_t u_t + & \Omega_{13} y_t y_{t-} + & \Omega_{14} y_t u_{t-1} + & \Omega_{15} y_t \\
+ & + & + & + & + \\
\Omega_{21} u_t y_t + & \Omega_{22} u_t^2 + & \Omega_{23} u_t y_{t-1} + & \Omega_{24} u_t u_{t-1} + & \Omega_{25} u_t \\
+ & + & + & + & + \\
\Omega_{31} y_{t-1} y_t + & \Omega_{32} y_{t-1} u_t + & \Omega_{33} y_{t-1}^2 + & \Omega_{34} y_{t-1} u_{t-1} + & \Omega_{35} y_{t-1} \\
+ & + & + & + & + \\
\Omega_{41} u_{t-1} y_t + & \Omega_{42} u_{t-1} u_t + & \Omega_{43} u_{t-1} y_{t-1} + & \Omega_{44} u_{t-1}^2 + & \Omega_{45} u_{t-1} \\
+ & + & + & + & + \\
\Omega_{51} y_t + & \Omega_{52} u_t + & \Omega_{53} y_{t-1} + & \Omega_{54} u_{t-1} + & \Omega_{55}
\end{bmatrix}
$$

$$J = (y_t - s_t)^2 + \omega u_t^2 + \lambda (u_t - u_{t-1})^2 =$$

$$= y_t^2 - 2y_t s_t + s_t^2 + \omega u_t^2 + \lambda u_t^2 - 2\lambda u_t u_{t-1} + \lambda u_{t-1}^2$$

$$\begin{array}{c} \\ y_t \\ u_t \\ y_{t-1} \\ u_{t-1} \\ 1 \end{array} \begin{array}{c} y_t, \quad u_t, \quad y_{t-1}, \quad u_{t-1}, \quad 1 \\ \begin{bmatrix} 1 & 0 & 0 & 0 & -s_t \\ 0 & \omega + \lambda & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & \lambda & 0 \\ -s_t & 0 & 0 & 0 & s_t^2 \end{bmatrix} \end{array}$$

Střední hodnota

$$E\left[x_t' x_t | u_t, d(t-1)\right] = E\left[(Mx_{t-1} + Nu_t + w_t)'(Mx_{t-1} + Nu_t + w_t)\right] =$$

$$= (Mx_{t-1} + Nu_t)'(Mx_{t-1} + Nu_t) + 2(Mx_{t-1} + Nu_t)' E[w_t] + E\left[w_t' w_t\right]$$

$$= (Mx_{t-1} + Nu_t)'(Mx_{t-1} + Nu_t) + E\left[w_t' w_t\right] =$$

$$\left(x_{t-1}' M' + u_t' N'\right)(Mx_{t-1} + Nu_t) + r =$$

$$= x_{t-1}' M' M x_{t-1} + 2x_{t-1}' M' N u_t + u_t' N' N u_t + r$$

Doplnění na čtverec

$$\text{formula} \quad (x+a)^2 = x^2 + 2xa + a^2$$

$$x^2 + xm + c = x^2 + 2x\frac{m}{2} + \left(\frac{m}{2}\right)^2 - \left(\frac{m}{2}\right)^2 + c =$$

$$= \underbrace{\left(x + \frac{m}{2}\right)^2}_{\text{square}} + \underbrace{c - \left(\frac{m}{2}\right)^2}_{\text{reminder}}$$

$\cdots$ ve vektorech

$$x'Ax + 2x'Bm + C = x'Ax + 2x'A\underline{A^{-1}m} + m'A^{-1}\underbrace{AA^{-1}}_{=E}m - m'A^{-1}\underbrace{AA^{-1}}_{=E}m + C =$$

$$= \underbrace{\left(x' + A^{-1}m\right)' A \left(x' + A^{-1}m\right)}_{=0 \ \to m_{\text{opt}}} + \underbrace{C - m'A^{-1}m}_{\text{zbytek}}$$