1 Random variable

1.1 Example

The density function is given by the formula

$$f(x) = 1 - |x - 2|$$

for $x \in (1, 3)$ otherwise 0. Determine distribution function.

1.2 Example

Determine 0.05-quantil of random variable with density function $f(x) = 3\left(x^2 + 2x + 5\right)/19$ for $x \in (0, 1)$ otherwise zero.

1.3 Example

Verify that the formula $5\exp\{-5x\}$ for $x \geq 0$ is a density function.

1.4 Example

Show that random variables X and Y with density function

$$f(x, y) = \left(\frac{\mathrm{e}}{\mathrm{e} - 1}\right)^2 \mathrm{e}^{(-x-y)}, \quad \text{on } (0, 1) \times (0, 1)$$

are independent.

1.5 Example

a) Show that random variables X and Y with probability function

$x\backslash y$	1	2
1	0.2	0.4
2	0.3	0.1

are dependent.

b) Change the second row of the probability function so that X and Y would become independent.

1.6 Example

Determine expectation $E[X]$ and $E[Y]$ for random variables X and Y with probability density function

$$f(x,y) = 2x\sin(y),\ \ x \in (0,1)\,,\ y \in (0,\pi/2)$$

1.7 Example

Find conditional probability function $f(x|y)$ to the joint

$x\backslash y$	1	2	3
1	0.1	0.3	0.2
2	0.2	0.1	0.1

1.8 Example

Random variable X has distribution function

$$F(x) = \frac{1}{4}x^2,\ \ x \in (0,2)$$

Determine the expectation $E[X]$

Solutions

Solution to 1.1

Draw the function.

The formula can be written in the following way

$$f(x) = \begin{cases} 0 & \text{pro } x < 1 \\ x - 1 & \text{pro } x \in (1, 2) \\ 3 - x & \text{pro } x \in (2, 3) \\ 0 & \text{pro } x > 3 \end{cases}$$

The distribution function is the following integral

$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$

We shall integrate on individual intervals:

For $x < 1$ it is $f(x) = 0$ and so $F(x) = 0$

For $x \in (1, 2)$ we have

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{1} f(t)\, dt + \int_{1}^{x} f(t)\, dt =$$

$$= 0 + \int_{1}^{x} (t - 1)\, dt = \left[\frac{1}{2}f^2 - t\right]_1^x = \frac{1}{2}x^2 - x - \frac{1}{2} + 1 = \frac{1}{2}x^2 - x + \frac{1}{2}$$

and for the interval $x \in (2, 3)$ we have: up to 1 it is 0, to 2 it is $F(x = 2) = \frac{1}{2}$ (substituted to the previous formula) and so

$$F(x) = 0 + \frac{1}{2} + \int_{2}^{x} (3 - t)\, dt = \frac{1}{2} + \left[3t - \frac{1}{2}t^2\right]_2^x =$$

$$= \frac{1}{2} + 3x - \frac{1}{2}x^2 - 6 + 2 = -\frac{1}{2}x^2 + 3x - \frac{7}{2}$$

Check - in $x = 3$ the distribution function ends, and so, here must be $F(3) = 1$ - OK.

For $x > 3$ the density is again $f(x) = 0$ and so

$$F(x) = 1.$$

Solution to 1.2

$$F(x) = \int_0^x 3\left(t^2+2t+5\right)/19dt = \frac{3}{19}\left(\frac{x^3}{3}+x^2+5x\right)$$

and now, it should hold

$$F(\zeta) = 0.05 \quad \rightarrow \quad \frac{3}{19}\left(\frac{\zeta^3}{3}+\zeta^2+5\zeta\right) = 0.05$$

which is the third order polynomial equation which cannot be simply solved. Must be solved numerically.

Solution to 1.3

It is nonnegative and

$$\int_0^\infty 0.5\exp\{-0.5x\}\,dx = -0.5\left[\frac{1}{0.5}\exp\{-0.5x\}\right]_0^\infty = 1$$

Solution to 1.4

Yes, they are.

$$f(x,y) = \left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^2 \mathrm{e}^{(-x-y)} = \frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-x}\cdot\frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-y} = f(x)\cdot f(y)$$

or by integration

$$f(y) = \int_0^1 \left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^2 \mathrm{e}^{(-x-y)}dx = \left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^2 \mathrm{e}^{-y}\int_0^2 \mathrm{e}^{-x}dx =$$

$$= -\left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^2 \mathrm{e}^{-y}\left[\mathrm{e}^{-x}\right]_0^1 = \left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^2 \mathrm{e}^{-y}\left(1-\mathrm{e}^{-1}\right) =$$

$$= \left(\frac{\mathrm{e}}{\mathrm{e}-1}\right)^2 \mathrm{e}^{-y}\frac{\mathrm{e}-1}{\mathrm{e}} = \frac{\mathrm{e}}{\mathrm{e}-1}\mathrm{e}^{-y}$$

and the same for $f(x)$.

Solution to 1.5

Solution to a)

Yes, they are dependent.

$$f(x) = [0.6, 0.4], \quad f(y) = [0.5,\ 0.5]$$

and e.g.

$$f(x=1, x=1) = 0.2 \neq 0.6\cdot 0.5 = 0.3 = f(x)\cdot f(y)$$

Solution to b)

The probability function will be

$x\backslash y$	1	2
1	0.2	0.4
2	a	b

and the marginals are $f(x) = [0.6,\ a+b]$ and $f(y) = [0.2+a,\ 0.4+b]$.

Now, it must hold

$$f(x{=}1)\, f(y=1) = 0.2$$

$$f(x=1)\, f(y=2) = 0.4$$

it is

$$0.6 \cdot (0.2 + a) = 0.2$$

$$0.6 \cdot (0.4 + b) = 0.4$$

$\rightarrow$

$$a = 2/15 \text{ and } b = 4/15$$

Or: the rows (or columns) must be linearly dependent and sum of all is 1. That is sum of the second column is 0.4. Then the second row is

$$q\,(0.2,\ 0.4) = 0.4$$

$$q = \frac{0.4}{0.6} = \frac{2}{3}$$

Then the second row is

$$\frac{2}{3}\,(0.2,\ 0.4) = \left(\frac{0.4}{3},\ \frac{0.8}{3}\right) = (2/15,\ 4/15)\,.$$

Solution to 1.6

Hint: $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$ where $f(x)$ is the marginal. Similarly for Y.

$$f(y) = \int_0^1 2x \sin(y)\, dx = \sin(y)\left[x^2\right]_0^1 = \sin(y)$$

$$f(x) = \int_0^{\pi/2} 2x \sin(y)\, dy = 2x\,[\cos(y)]_0^{\pi/2} = 2x$$

$$E[Y] = \int_0^{\pi/2} y \cdot \sin(y)\, dy = \ (*)$$

$$|y = u,\, \sin(y) = v';\; u' = 1,\, v = \cos(y)\,|$$

$$(*) = [uv]_0^{\pi/2} - \int_0^{\pi/2} u'v = [y\cos(y)]_0^{\pi/2} - \int_0^{\pi/2} \cos(y)\,dy =$$
$$= 0 - \left(- [\sin(y)]_0^{\pi/2}\right) = 1$$

$$E[X] = \int_0^1 x \cdot 2x dx = \left[\frac{2}{3}x^3\right]_0^1 = \frac{2}{3}$$

Solution to 1.7

Hint: Use definition of conditional probability function or (which is the same) normalize columns to the sum equal to one.

$x\backslash y$	1	2	3
1	0.1/0.3	0.3/0.4	0.2/0.3
2	0.2/0.3	0.1/0.4	0.1/0.3

=

$x\backslash y$	1	2	3
1	1/3	3/4	2/3
2	2/3	1/4	1/3

Solution to 1.8

$$f(x) = \frac{1}{2}x,\;\; x\,(0,2)$$

$$E[X] = \int_0^2 x \cdot \frac{1}{2}x dx = \frac{1}{6}\left[x^3\right]_0^2 = \frac{4}{3}$$