Matrices

1) Exercise: Calculate the products AB a BA (if exist)

a) $A = \begin{pmatrix} 2 & 5 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ -4 & 6 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 2 & 5 \\ -3 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -3 \\ -4 & 6 \end{pmatrix} = \begin{pmatrix} 2-20, & -6+30 \\ -3-12, & -9+18 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -18 & 24 \\ -15 & 9 \end{pmatrix}}}$$

$$2\times\underbrace{2 \quad = \quad 2}\times 2 \qquad\qquad 2\times 2$$

$$B \cdot A = \begin{pmatrix} 1 & -3 \\ -4 & 6 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 2+9, & 5-9 \\ -8-18, & -20+18 \end{pmatrix} = \underline{\begin{pmatrix} 11 & -4 \\ -26 & -2 \end{pmatrix}}$$

$$2\times\underbrace{2 \qquad 2}\times 2 \qquad\qquad 2\times 2$$

b) $A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 2 & -1 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 3 \\ -5 & 1 \\ 2 & -3 \\ 5 & 2 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 2 & -1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -5 & 1 \\ 2 & -3 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 5+10+6+5, & 3-2-9+2 \\ 10+5+0+25, & 6-1+0+10 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 26 & -6 \\ 40 & 15 \end{pmatrix}}}$$

$$2\times\underbrace{4 \qquad\qquad 4}\times 2 \qquad\qquad 2\times 2$$

$$B \cdot A = \begin{pmatrix} 5 & 3 \\ -5 & 1 \\ 2 & -3 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 & 1 \\ 2 & -1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 5+6, & -10-3, & 15+0, & 5+15 \\ -5+2, & 10-1, & -15+0, & -5+5 \\ 2-6, & -4+3, & 6+0, & 2-15 \\ 5+4, & -10-2, & 15+0, & 5+10 \end{pmatrix} =$$

$$4\times\underbrace{2 \qquad 2}\times 4 \qquad\qquad 4\times 4$$

$$= \underline{\underline{\begin{pmatrix} 11 & -13 & 15 & 20 \\ -3 & 9 & -15 & 0 \\ -4 & -1 & 6 & -13 \\ 9 & -12 & 15 & 15 \end{pmatrix}}}$$

Try to calculate c, d

2) Exercise: Calculate AB - BA (if exists)

a) $A = \begin{pmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -1+10+4, & 1+15-8, & 1+5-2 \\ 0+4+2, & 0+6-4, & 0+2-1 \\ 3+2-2, & -3+3+4, & -3+1+1 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 13 & 8 & 4 \\ 6 & 2 & 1 \\ 3 & 4 & -1 \end{pmatrix}}$$

$$BA = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1+0-3, & -5+2+1, & 2-1+1 \\ 2+0-3, & 10+6+1, & -4-3+1 \\ -2+0-3, & -10+8+1, & 4-4+1 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} -4 & -2 & 2 \\ -1 & 17 & -6 \\ -5 & -1 & 1 \end{pmatrix}}$$

$$AB - BA = \begin{pmatrix} 13 & 8 & 4 \\ 6 & 2 & 1 \\ 3 & 4 & -1 \end{pmatrix} - \begin{pmatrix} -4 & -2 & 2 \\ -1 & 17 & -6 \\ -5 & -1 & 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 17 & 10 & 2 \\ 7 & -15 & 7 \\ 8 & 5 & -1 \end{pmatrix}}}$$

Try to calculate b.

3) Exercise: Calculate

a) $\begin{pmatrix} 4 & 2 & 1 \\ 3 & -2 & -2 \\ 1 & 0 & 5 \end{pmatrix}^2$

$$\begin{pmatrix} 4 & 2 & 1 \\ 3 & -2 & -2 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ 3 & -2 & -2 \\ 1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 16+6+1, & 8-4+0, & 4-4+5 \\ 12-6-2, & 6+4+0, & 3+4-10 \\ 4+0+5, & 2+0+0, & 1+0+25 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 23 & 4 & 5 \\ 4 & 10 & -3 \\ 9 & 2 & 26 \end{pmatrix}}}$$

Try to calculate b, c.

d) $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 1 & -3 \end{pmatrix}^4$

$A^2 = A \cdot A$ $\qquad B = A^2$

$A^4 = (A \cdot A)^2 = B \cdot B$

$$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 1 & -3 \end{pmatrix} = \begin{pmatrix} 1+0+4 & 0+0+2 & 2+0-6 \\ -1-1+0 & 0+1+0 & -2+0-0 \\ 2-1-6 & 0+1-3 & 4+0+9 \end{pmatrix} = \begin{pmatrix} 5 & 2 & -4 \\ -2 & 1 & -2 \\ -5 & -2 & 13 \end{pmatrix}$$

$$A^2 \cdot A^2 = \begin{pmatrix} 5 & 2 & -4 \\ -2 & 1 & -2 \\ -5 & -2 & 13 \end{pmatrix} \begin{pmatrix} 5 & 2 & -4 \\ -2 & 1 & -2 \\ -5 & -2 & 13 \end{pmatrix} = \begin{pmatrix} 25-4+20 & 10+2+8 & 20-4-52 \\ -10-2+10 & -4+1+4 & 8-2-26 \\ -25+4-65 & -10-2-26 & 20+4+169 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 41 & 20 & -36 \\ -2 & 1 & -20 \\ -86 & -36 & 193 \end{pmatrix}}}$$

e) $\begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix}^3$

$A^2 = A \cdot A$

$A^3 = A \cdot A^2$

$$A^2 = \begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 4-12 & 6+15 \\ -8-20 & -12+25 \end{pmatrix} = \begin{pmatrix} -8 & 21 \\ -28 & 13 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -8 & 21 \\ -28 & 13 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} -16-84 & -24+105 \\ -56-52 & -84+65 \end{pmatrix} = \underline{\begin{pmatrix} -100 & 81 \\ -108 & -19 \end{pmatrix}}$$

f) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{129}$

$A^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

$A^4 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

$\vdots$

$$A^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \longrightarrow A^{m+1} = \begin{pmatrix} 1 & m+1 \\ 0 & 1 \end{pmatrix} \ ?$$

$$A^{m+1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m+1 \\ 0 & 1 \end{pmatrix} \text{ OK} \Longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{129} = \underline{\begin{pmatrix} 1 & 129 \\ 0 & 1 \end{pmatrix}}$$

4) Exercise: Calculate the products AA^T and A^TA.

a) $A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ -3 & 1 & 1 \end{pmatrix}$

$$A \cdot A^T = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & -3 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+4+1, & -2+6+0, & -3+2+1 \\ -2+6+0, & 4+9+0, & 6+3+0 \\ -3+2+1, & 6+3+0, & 9+1+1 \end{pmatrix} =$$

$$= \begin{pmatrix} 6 & 4 & 0 \\ 4 & 13 & 9 \\ 0 & 9 & 11 \end{pmatrix}$$

$$A^T \cdot A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ -3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+4+9, & 2-6-3, & 1+0-3 \\ 2-6-3, & 4+9+1, & 2+0+1 \\ 1+0-3, & 2+0+1, & 1+0+1 \end{pmatrix} =$$

$$= \begin{pmatrix} 14 & -4 & -2 \\ -7 & 14 & 3 \\ -2 & 3 & 2 \end{pmatrix}$$

b) $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 1 \end{pmatrix}$

$$A \cdot A^T = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1+4+1, & 3+10-1 \\ 3+10-1, & 9+25+1 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ 12 & 35 \end{pmatrix}$$

$$A^T \cdot A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1+9, & 2+15, & -1+3 \\ 2+15, & 4+25, & -2+5 \\ -1+3, & -2+5, & 1+1 \end{pmatrix} = \begin{pmatrix} 10 & 17 & 2 \\ 17 & 29 & 3 \\ 2 & 3 & 2 \end{pmatrix}$$

Try to calculate c, d, e

5) Exercise: Calculate the row rank of matrices

a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 3 & 2 \end{pmatrix}$ the type 4×2

- Without a calculation, we see rank A is 2.
- With a calculation: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \\ 3 & 2 \end{pmatrix} -1 + \; -3 + \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \\ 0 & -1 \end{pmatrix} + + \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ \boxed{0 \;\; 0} \\ \boxed{0 \;\; 0} \end{pmatrix} \Longrightarrow$ rank A = 2

2 LI vectors

5)

b) $\begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix}$ the type 4x4

$$\begin{pmatrix} 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 1 & 3 & 4 & -2 \\ 4 & -3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & -2 \\ 2 & 1 & 3 & -1 \\ 3 & -1 & 2 & 0 \\ 4 & -3 & 1 & 1 \end{pmatrix} \overset{-2\;-3\;-4}{\sim} \begin{pmatrix} 1 & 3 & 4 & -2 \\ 0 & -5 & -5 & 3 \\ 0 & -10 & -10 & +6 \\ 0 & -15 & -15 & 9 \end{pmatrix} \overset{-2\;-3}{\sim} \begin{pmatrix} 1 & 3 & 4 & -2 \\ 0 & -5 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 L.I vectors

rank A = 2

c) $\begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$ · the type 4x5

$$\begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \overset{-1\,+}{\sim} \begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} \overset{-1\;-1\,+\,+}{\sim} \begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3 L.I vectors

rank A = 3

Try to calculate d, e, f, g, h, i, j.

6) Exercise: Calculate the matrix A^{-1} (if exists).

a) $A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \overset{-1\,+\,+}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \overset{+}{\sim} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 \end{array}\right) \overset{2\,+}{\sim}$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 & 2 & 1 & 1 \end{array}\right) \overset{3\;-3\,+\;-3\,+\,+}{\sim} \left(\begin{array}{cccc|cccc} 3 & 3 & 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 3 & -3 & 0 & -1 & 2 & -2 & 1 \\ 0 & 0 & 3 & 0 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 3 & -1 & 2 & 1 & 1 \end{array}\right) \overset{+}{\sim} \left(\begin{array}{cccc|cccc} 3 & 3 & 0 & 0 & 2 & 2 & 1 & 1 \\ 0 & 3 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 3 & -1 & 2 & 1 & 1 \end{array}\right) \overset{-1\,+}{\sim}$$

rank A = 4

$$\left(\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 1 & 2 & -1 \\ 0 & 3 & 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 3 & -1 & 2 & 1 & 1 \end{array}\right) \qquad A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 1 & 1 & -1 & 2 \\ 2 & -1 & 1 & 1 \\ -1 & 2 & 1 & 1 \end{pmatrix}$$

c)

b) $A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$ $(A|E) \rightsquigarrow (E|A^{-1})$

$$\underbrace{\left(\begin{array}{ccc|ccc} 3 & 2 & 0 & 1 & 0 & 0 \\ 2 & 4 & -2 & 0 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right)}_{A \quad E} \overset{-2,\ 3)+}{\sim} \left(\begin{array}{ccc|ccc} 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 8 & -6 & -2 & 3 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right) \underset{4)+}{\sim} \left(\begin{array}{ccc|ccc} 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 8 & -6 & -2 & 3 & 0 \\ 0 & 0 & 14 & -2 & 3 & 4 \end{array}\right) \overset{7}{\underset{3)+}{\sim}}$$

rank $A = 3 \longrightarrow A^{-1}$ exists

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 56 & 0 & -20 & 30 & 12 \\ 0 & 0 & 14 & -2 & 3 & 4 \end{array}\right) \overset{-28)+}{\sim} \left(\begin{array}{ccc|ccc} -84 & 0 & 0 & -48 & 30 & 12 \\ 0 & 56 & 0 & -20 & 30 & 12 \\ 0 & 0 & 14 & -2 & 3 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{48}{84} & \frac{-30}{84} & -\frac{12}{84} \\ 0 & 1 & 0 & \frac{-20}{56} & \frac{30}{56} & \frac{12}{56} \\ 0 & 0 & 1 & \frac{-2}{14} & \frac{3}{14} & \frac{4}{14} \end{array}\right)$$

the diagonal matrix

$\underbrace{}_{E}$ $\underbrace{}_{A^{-1}}$

$$A^{-1} = \frac{1}{14} \begin{pmatrix} \frac{48}{6} & -\frac{30}{6} & -\frac{12}{6} \\ -\frac{20}{4} & \frac{30}{4} & \frac{12}{4} \\ -2 & 3 & 4 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 8 & -5 & -2 \\ -\frac{10}{2} & \frac{15}{2} & \frac{6}{2} \\ -2 & 3 & 4 \end{pmatrix}$$

$$= \frac{1}{28} \begin{pmatrix} 16 & -10 & -4 \\ -10 & 15 & 6 \\ -4 & 6 & 8 \end{pmatrix}$$

c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$ $(A|E) \rightsquigarrow (E|A^{-1})$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \overset{-2)+\ -1)+}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \overset{-1)+}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \underset{-1)+}{\sim}$$

rank $A = 3 \longrightarrow A^{-1}$ exists

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \overset{)+}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \overset{-1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

the diagonal matrix

$\underbrace{}_{E}$ $\underbrace{}_{A^{-1}}$

$$A^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

! the numerical control $A \cdot A^{-1} = E$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ OK

d) $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

A is the rectangular matrix, the type is 3x4

$\longrightarrow$ A^{-1} does not exist

Try to calculate e, f, g, h, i, j

7) Exercise: Solve the matrix equation for a matric X.

a) $$X \cdot \begin{pmatrix} 3 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 2 & 7 \\ 8 & 6 & -5 \end{pmatrix}$$

Our preposition

$$X \cdot A = B \qquad | \cdot A^{-1}, \text{ from the right side}$$
$$X \cdot \underbrace{A A^{-1}} = B \cdot A^{-1}$$
$$X \cdot E = B \cdot A^{-1}$$
$$\underline{\underline{X = B \cdot A^{-1}}}$$

<u>1. step: A^{-1}</u>

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \begin{matrix} -3)+ \\ -1 \\ )+ \end{matrix} \sim \left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 6 & 1 & 1 & -3 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \end{array}\right) \begin{matrix} \\ \sim)+ \\ -6 \end{matrix} \left(\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 6 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 & 3 & -6 \end{array}\right) \begin{matrix})+ \\ )+ \\ -1 \;\; -1 \end{matrix}$$

the rank A = 3 $\longrightarrow$ A^{-1} exists

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & -3 & 6 \\ 0 & 6 & 0 & 0 & -6 & 6 \\ 0 & 0 & 1 & 1 & 3 & -6 \end{array}\right) \sim \left(\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{E} \middle| \underbrace{\begin{array}{ccc} 0 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & 3 & -6 \end{array}}_{A^{-1}}\right)$$

he diagonal matrix

$$A^{-1} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & 3 & -6 \end{pmatrix}$$

2. step

$$X = B \cdot A^{-1} = \begin{pmatrix} 4 & 5 & 6 \\ -1 & 2 & 7 \\ 8 & 6 & -5 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 0 & -1 & 1 \\ 1 & 3 & -6 \end{pmatrix} = \begin{pmatrix} 0+0+6, & -4-5+18, & 8+5-36 \\ 0+0+7, & 1-2+21, & -2+2-42 \\ 0+0-5, & -8-6-15, & 16+6+30 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 6 & 9 & -23 \\ 7 & 20 & -42 \\ -5 & 29 & 52 \end{pmatrix}}}$$

b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 5 \\ 7 & 1 \end{pmatrix}$

Our preposition

$A \cdot X = B \quad | \; A^{-1}$, from the left side

$\underbrace{A^{-1} \cdot A}_{} \cdot X = A^{-1} \cdot B$

$E \cdot X = A^{-1} \cdot B$

$\underline{\underline{X = A^{-1} \cdot B}}$

1. step : A^{-1}

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right)^{-3} \!\! \sim \!\big)+ \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right)\!\big)^{+} \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{E} \qquad \underbrace{\begin{matrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{matrix}}_{A^{-1}}$$

the rank A = 2
$\longrightarrow A^{-1}$ exists

the diagonal matrix

$$\underline{\underline{A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} -4 & +2 \\ 3 & -1 \end{pmatrix}}}$$

2. step:

$$X = A^{-1} \cdot B = \frac{1}{2} \cdot \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 7 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -4+14, & -20+2 \\ 3-4, & 15-1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 10 & -18 \\ -4 & 14 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 5 & -9 \\ -2 & 7 \end{pmatrix}}}$$

the control:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & -9 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 5-4, & -9+14 \\ 15-8, & -27+28 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 7 & 1 \end{pmatrix} = B$$

OK

Try to solve c.

d) $\begin{pmatrix}2&4&0\\1&2&0\\0&0&1\end{pmatrix} \cdot X = \begin{pmatrix}1&2&4\\-5&-1&1\\-1&1&0\end{pmatrix}$

$A \cdot X = B$ $\qquad$ $A = \begin{pmatrix}2&4&0\\1&2&0\\0&0&1\end{pmatrix}$

$$\begin{pmatrix}2&4&0\\1&2&0\\0&0&1\end{pmatrix} \overset{-\frac{1}{2}) +}{\sim} \begin{pmatrix}1&2&0\\0&0&0\\0&0&1\end{pmatrix} \updownarrow \sim \begin{pmatrix}1&2&0\\0&0&1\\0&0&0\end{pmatrix}$$

$\underline{\underline{\text{rank } A = 2}}$, A^{-1} does not exists

$$(A|B): \left(\begin{array}{ccc|ccc}2&4&0&1&2&4\\1&2&0&-5&-1&1\\0&0&1&-1&1&0\end{array}\right) \overset{-2) +}{\sim} \left(\begin{array}{ccc|ccc}2&4&0&1&2&4\\0&0&0&11&0&2\\0&0&1&-1&1&0\end{array}\right) \updownarrow \sim$$

$$\left(\begin{array}{ccc|ccc}2&4&0&1&2&4\\0&0&1&-1&1&0\\0&0&0&11&0&2\end{array}\right)$$

$\underline{\underline{\text{rank } (A|B) = 3}}$

the necessary condition: rank A = rank $(A|B)$

$\longrightarrow 2 \neq 3$

$\Longrightarrow$ the equation has $\underline{\underline{\text{no}}}$ solution

Try to solve e, f.