System of linear equations

1) Exercise: Solve the system of linear equations

a)
$$\begin{aligned} 2x - y - z &= 4 \\ 3x + 4y - 2z &= 2 \\ 3x - 2y + 4z &= 11 \end{aligned}$$

- nonhomogenous system
- three equations
- three unknowns

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 3 & 4 & -2 & 2 \\ 3 & -2 & 4 & 11 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 0 & 11 & -1 & -8 \\ 0 & -6 & 6 & 9 \end{array}\right)\frac{1}{3} \updownarrow \sim \left(\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 0 & -2 & 2 & 3 \\ 0 & 11 & -1 & -8 \end{array}\right) \sim$$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 4 \\ 0 & -2 & 2 & 3 \\ 0 & 0 & 20 & 17 \end{array}\right) \quad \begin{matrix} \circ\circ\circ \\ \circ\circ \\ \circ \end{matrix} \longrightarrow \begin{aligned} h(A) &= h(A|B) = 3 \\ n &= 3 \\ \dim V &= 3 - 3 = 0 \end{aligned}$$

$\circ$ $20z = 17$
$$z = \frac{17}{20}$$

$\circ\circ$ $-2y + 2z = 3$
$$2y = 2z - 3 = \frac{17}{10} - 3 = \frac{17-30}{10} = -\frac{13}{10}$$
$$y = -\frac{13}{20}$$

$\circ\circ\circ$ $2x - y - z = 4$
$$2x = 4 + y + z = 4 - \frac{13}{20} + \frac{17}{20} = 4 + \frac{1}{5} = \frac{21}{5}$$
$$x = \frac{21}{10}$$

Solution
$$\left[\frac{21}{10}, -\frac{13}{20}, \frac{17}{20}\right] + \{(0,0,0)\} = \left[\frac{21}{10}, -\frac{13}{20}, \frac{17}{20}\right]$$

b)
$$\begin{aligned} (i+1)x + (1-i)y + (1+i)z &= 1 \\ (1-i)x + (1+3i)y + (i-1)z &= 0 \\ x + (1+i)y + iz &= 1 \end{aligned}$$

- nonhomogenous system
- three equations
- three unknowns

complex numbers: $i \cdot i = -1$

$$
\left(\begin{array}{ccc|c} 1 & 1+i & i & 1 \\ 1+i & 1-i & 1+i & 1 \\ 1-i & 1+3i & -1+i & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1+i & i & 1 \\ 0 & 1-3i & 2 & -i \\ 0 & 1+3i & -2 & -1+i \end{array}\right) \sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1+i & i & 1 \\ 0 & 1-3i & 2 & -i \\ 0 & 2 & 0 & -1 \end{array}\right) \end{array} \begin{array}{l} \circ\circ\circ \\ \circ\circ \\ \circ \end{array}
$$

$h(A) = h(A|b) = 3$

$n = 3$

$\dim V = 3-3=0$

$\circ$ $2y = -1$

$\underline{\underline{y = -\frac{1}{2}}}$

$\circ\circ$ $(1-3i)y + 2z = -i$

$(1-3i)(-\frac{1}{2}) + 2z = -i$

$-\frac{1}{2} + \frac{3}{2}i + 2z = -i$

$2z = \frac{1}{2} - \frac{5}{2}i$

$\underline{\underline{z = \frac{1}{4} - \frac{5}{4}i}} = \frac{1}{4}(1-5i)$

$\circ\circ\circ$ $x + (1+i)y + iz = 1$

$x = 1 - iz - (1+i)y = 1 - i(\frac{1}{4} - \frac{5}{4}i) + \frac{1}{2}(1+i) =$

$= 1 - \frac{1}{4}i - \frac{5}{4} + \frac{1}{2} + \frac{1}{2}i = \frac{3}{2} - \frac{5}{4} + \frac{1}{4}i = \frac{1}{4} + \frac{1}{4}i = \frac{1}{4}(1+i)$

Solution: $\underline{\underline{\left[\frac{1}{4}(1+i), -\frac{1}{2}, \frac{1}{4}(1-5i)\right]}}$

Try to solve c, d, e.

f)
$$
\begin{aligned}
2x - y + 3z &= 0 \\
x + 3y + 2z &= 0 \\
3x - 5y + 4z &= 0 \\
x + 17y + 4z &= 0
\end{aligned}
$$

- homogenous system
- three equations
- three unknowns

$$
\begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 3 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{pmatrix} \sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 2 & -1 & 3 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \circ\circ \\ \circ \\ \\ \\ \end{array}
$$

$h(A) = 2$

$n = 3$

$\dim V = 3-2 = 1$

$\circ$ $-7y - z = 0$ $\quad z = 7 \quad y = -1$

$\circ\circ$ $2x - y + 3z = 0$ $\quad 2x = y - 3z = -1-21$

$x = -11$

Solution: $\underline{\{(-11, -1, 7)\}}$

my choose (everything, not zero!)

Try to solve g, h.

i) $$\begin{aligned} x+3y+2z&=2\\ 2x-y+3z&=7\\ 3x-5y+4z&=12\\ x+17y+4z&=-4 \end{aligned}$$

- nonhomogenous system
- 4 equations
- 3 unknowns

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2\\ 2 & -1 & 3 & 7\\ 3 & -5 & 4 & 12\\ 1 & 17 & 4 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 2\\ 0 & -7 & -1 & 3\\ 0 & -14 & -2 & 6\\ 0 & 14 & 2 & -6 \end{array}\right) \sim \begin{array}{c} \begin{array}{ccc} x & y & z \end{array}\\ \left(\begin{array}{ccc|c} 1 & 3 & 2 & 2\\ 0 & -7 & -1 & 3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$h(A)=2$
$h(A|B)=2$] $\Rightarrow$
$\dim V = 3-2=1$

a) homogenous part

$$\begin{pmatrix} 1 & 3 & 2\\ 0 & -7 & -1 \end{pmatrix}$$

$\{(-11, -1, 7)\}$

$0 - 7y - z = 0 \quad z = 7$

$y=-1$

$x+3y+2z=0$

$x=-2z-3y=-14+3=-11$

b) nonhomogenous part

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array}\\ \left(\begin{array}{ccc|c} 1 & 3 & 2 & 2\\ 0 & -7 & -1 & 3 \end{array}\right) \end{array}$$

$[\frac{12}{7}, \frac{-4}{7}, 1]$

$0-7y-z=3 \quad z=1$

$7y=-3-z=-3-1=-4$

$y=\frac{-4}{7}$

$x+3y+2z=2$

$x=2-2z-3y=2-2+3\frac{4}{7}=\frac{12}{7}$

Solution: $[\frac{12}{7}, \frac{-4}{7}, 1] + \{(-11,-1,7)\}$

Try to solve j, k, l, m, n, o, p, q, r, s, t, u, v.

2) Exercise: Solve the system of linear equations

a)

$$\begin{aligned} x + 2y \quad\quad &= -1 \\ y + z \quad &= 0 \\ x \quad\quad - u &= -1 \\ x + y - z + u &= 2 \end{aligned}$$

- nonhomogenous system
- 4 equations
- 4 unknowns

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 \\ 1 & 1 & -1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 & 3 \end{array}\right) \sim \begin{array}{c} \begin{array}{cccc} x & y & z & u \end{array} \\ \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right) \end{array} \begin{array}{l} 0000 \\ 000 \\ 00 \\ 0 \end{array} \uparrow$$

$h(A) = h(A|B) = 4$

$n = 4$

$\dim V = 4 - 4 = 0$

a) nonhomogenous part

0 $\quad u = 3$

00 $\quad 2z - u = 0$

$$z = \frac{1}{2}u = \frac{3}{2}$$

000 $\quad y + z = 0$

$$y = -z = -\frac{3}{2}$$

0000 $\quad x + y = -1$

$$x = -1 - y = -1 + \frac{3}{2} = \frac{1}{2} \qquad \left[\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, 3\right]$$

Solution: $\quad \left[\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, 3\right] + \{(0,0,0,0)\} = \left[\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, 3\right]$

Try to solve b, c, d, e, f, g, h, i

j)

$$\begin{aligned} ix + y \quad\quad &= -1 \\ 3x + 3y - z &= 0 \\ 2x - y - 2z &= -4 + 3i \end{aligned}$$

- nonhomogenous system
- 3 equations
- 3 unknowns
- complex numbers

$$\left(\begin{array}{ccc|c} i & 1 & 0 & 1 \\ 3 & 3 & -1 & 0 \\ 2 & -1 & -2 & -4+3i \end{array}\right) \begin{array}{c} -3)+ \\ i \ \ -2 \\ 3) \end{array} \sim \left(\begin{array}{ccc|c} i & 1 & 0 & 1 \\ 0 & -3+3i & -i & -3 \\ 0 & -9 & -4 & -12+9i \end{array}\right) \updownarrow \sim \left(\begin{array}{ccc|c} i & 1 & 0 & 1 \\ 0 & -9 & -4 & -12+9i \\ 0 & -3+3i & -i & -3 \end{array}\right) \begin{array}{c} 5) \\ 1+i \\ -3) \end{array}$$

$$\begin{array}{cccccc} x & y & z & & & \\ \end{array}$$

$$\left(\begin{array}{ccc|c} i & 1 & 0 & 1 \\ 0 & -9 & -4 & -12+9i \\ 0 & 0 & -4-i & -12-3i \end{array}\right) \begin{array}{l} \circ\circ\circ \\ \circ\circ \\ \circ \end{array}$$

$h(A) = h(A|b) = 3$

$n = 3$

$\dim V = 3 - 3 = 0$

$\circ \quad (-4-i)\, z = -12-3i$

$$z = \frac{12+3i}{4+i} \cdot \frac{4-i}{4-i} = \frac{48+12i-12i+3}{17} = \underline{\underline{\frac{51}{17}}}$$

$\circ\circ \quad -9y - 4z = -12+9i$

$$-9z = -12+9i-4z = -12+9i - \frac{204}{17} = \underline{\underline{9i}}$$

$\circ\circ\circ \quad ix + y = 1$

$ix + 9i = 1$

$ix = 1 - 9i$

$$x = \frac{1-9i}{i} \cdot \frac{i}{i} = \frac{i+9}{-1} = \underline{\underline{-9-i}}$$

Solution: $\left[-9-i,\ 9i,\ \frac{51}{17}\right] + \{(0,0,0)\} = \underline{\underline{\left[-9-i,\ 9i,\ \frac{51}{17}\right]}}$

Try to solve b.

3) Exercise: Solve the system of linear equations ($\underline{a}$ is a real parameter)

a) $4x + y + 2z = 0$

$x + ay - z = 0$

$6x + y + 2az = 0$

- homogenous system
- 3 equations
- 3 unknowns
- 1 parameter

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & a & -1 \\ 6 & 1 & 2a \end{pmatrix} \sim \begin{array}{c} + \\ -4) \end{array} \begin{array}{c} 3 \\ )+ \\ -2 \end{array} \begin{pmatrix} 4 & 1 & 2 \\ 0 & 1-4a & 6 \\ 0 & 1 & 6-4a \end{pmatrix} \updownarrow \sim \begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 6-4a \\ 0 & 1-4a & 6 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 6-4a \\ 0 & 0 & 20a \end{pmatrix}$$

$20a = 0 \rightsquigarrow a = 0$

I. $a = 0$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \circ\circ \\ \circ \end{array}$$

$h(A) = 2$
$m = 3$
$\dim V = 3 - 2 = 1$

$\circ \quad y + 6z = 0 \quad \underline{z = 1}$ (our choose)
$\underline{y = -6}$

$\circ\circ \quad 4x + y + 2z = 0$
$4x = -y - 2z = 6 - 2 = 4$
$x = 1;$

Solution: $\underline{\{(1, -6, 1)\}}$

II. $a \neq 0$

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 6-4a \\ 0 & 0 & 20a \end{pmatrix}$$

$h(A) = 3$
$m = 3$
$\dim V = 3 - 3 = 0$

Solution: $\underline{\{(0, 0, 0)\}}$

Try to solve b, c, d, e

f) $ax + y + z + u = 1$
$x + ay + z + u = a$
$x + y + az + u = a^2$

- nonhomogenous system
- 3 equations
- 3 unknowns
- 1 parameter

$$\left(\begin{array}{cccc|c} a & 1 & 1 & 1 & 1 \\ 1 & a & 1 & 1 & a \\ 1 & 1 & a & 1 & a^2 \end{array}\right) \begin{array}{l} -a)+ \\ -1 \\ \quad)+ \end{array} \sim \left(\begin{array}{cccc|c} 1 & a & 1 & 1 & a \\ 0 & 1-a^2 & 1-a & 1-a & 1-a^2 \\ 0 & 1-a & a-1 & 0 & a(1-a) \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & a & 1 & 1 & a \\ 0 & 1-a & a-1 & 0 & a(1-a) \\ 0 & 0 & a^2+a-2 & a-1 & (1-a)^2(-1-a) \end{array}\right)$$

$1-a$

$a \neq 0$
not necessary

$\nabla \; 1 - a \neq 0$
$\circ \quad a \neq 1$

I. $a = 1$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$h(A) = 1$
$m = 4$
$\dim V = 4 - 1 = 3$

Solution: $\underline{[0, 0, 0, 1] + \{(-1, 0, 0, 1), (-1, 0, 1, 0), (-1, 1, 0, 0)\}}$

II. $a^2+a-2=0 \quad a_{1,2}=\frac{-1\pm\sqrt{1+8}}{2} = \begin{cases} -2 \\ 1 \end{cases}$ ✓ 4)

$a=-2$

$$\begin{array}{cccc|c} x & y & z & u & \\ 1 & -2 & 1 & 1 & -2 \\ 0 & 3 & -3 & 0 & -6 \\ 0 & 0 & 0 & 3 & 9 \end{array}$$

$h(A)=3$
$n=4$
$\dim V = 4-3$

Solution: $[-7, 0, 2, 3] + \{(1,1,1,0)\}$

III. $a \neq 1 \wedge a \neq -2$

$$\left(\begin{array}{cccc|c} 1 & a & 1 & 1 & a \\ 0 & 1-a & a-1 & 0 & a(1-a) \\ 0 & 0 & (a-1)(a-2) & a-1 & (1-a)^2(-1-a) \end{array}\right)$$

Solution: $[a-1, a, 0, (1-a)(1+a)] + \{(2a-1, -1, -1, -a+2)\}$

g) $2x+3y-z=0$
$ax+4y+2z=0$

- homogenous system
- 2 equations
- 3 unknowns
- 1 parametr

$$\begin{pmatrix} 2 & 3 & -1 \\ a & 4 & 2 \end{pmatrix} \overset{-a}{\underset{2}{\sim}} \begin{pmatrix} 2 & 3 & -1 \\ 0 & 8-3a & 4+a \end{pmatrix}$$

? $8-3a=0$ and $4+a=0$

$a=\frac{8}{3}$ and $a=-4$ (it is not possible)

$\longrightarrow \forall a \in \mathbb{R} \quad h(A)=2 \quad n=3 \quad \dim V = 3-2=1$

I. $a=-4$ $\begin{pmatrix} 2 & 3 & -1 \\ 0 & 20 & 0 \end{pmatrix}$

Solution: $\{(\frac{1}{2}, 0, 1)\}$

II. $a=\frac{8}{3}$ $\begin{pmatrix} 2 & 3 & -1 \\ 0 & 0 & \frac{20}{3} \end{pmatrix}$

Solution: $\{(-3, 2, 0)\}$

III. $a \neq -4 \wedge a \neq \frac{8}{3}$

$(8-3a)y + (4+a)z = 0 \qquad \underline{z=1}$

$$\underline{y = -\frac{4+a}{8-3a}}$$

$$2x - \frac{12+3a}{8-3a} - 1 = 0$$

$$\underline{x = \left[1 + \frac{12+3a}{8-3a}\right] \cdot \frac{1}{2}}$$

Solution: $\underline{\left\{\left(\frac{1}{2}\left(1+\frac{12+3a}{8-3a}\right), \frac{-4-a}{8-3a}, 1\right)\right\}}$

Try to solve h, j.

4) Exercise: Solve the system of linear equations (a, b are real parameters)

a) $ax + by + z = 1$
$x + aby + z = b$
$x + by + az = 1$

- nonhomogenous system
- 3 equations
- 3 unknowns
- 2 parameters

Efective methode

$$\det A = \begin{vmatrix} a & b & 1 \\ 1 & ab & 1 \\ 1 & b & a \end{vmatrix} = a^3b + 2b - 3ab^2 = ab[a^2-3] + 2b = b[a^3-3a+2] =$$
$$= b[(a-1)^2(a+2)]$$

I. $a = 1 \wedge b = 0$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{No solution.}$$

$h(A) = 1 \quad h(A|b) = 2$

$1 \neq 2$

II. $a = -2 \wedge b = 0$

$$\left(\begin{array}{ccc|c} -2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$h(A) = 2$
$h(A|b) = 2$
$n = 3$
$\dim V = 3 - 2 = 1$

Solution: $\left[-\frac{1}{3}, 0, \frac{1}{3}\right] + \{(0,1,0)\}$

III. $a=1 \wedge b \neq 0$

$$\left(\begin{array}{ccc|c} 1 & b & 1 & 1 \\ 1 & b & 1 & b \\ 1 & b & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & b & 1 & 1 \\ 0 & 0 & 0 & b-1 \end{array}\right)$$

$b \neq 1$ ... $h(A)=1$ $\quad h(A|b)=2$ $\quad 1 \neq 2$

$\longrightarrow$ No solution.

$b=1$ $\quad (1\ 1\ 1\ |\ 1)$ $\quad h(A)=h(A|b)=1$

$n=3$ $\quad \dim V = 3-1 = 2$

Solution: $[1,0,0] + \{(-1,1,0),(-1,0,1)\}$

IV. $a=-2 \wedge b \neq 0$

$$\left(\begin{array}{ccc|c} -2 & b & 1 & 1 \\ 1 & -2b & 1 & b \\ 1 & b & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & b & 1 & 1 \\ 0 & 5b & -1 & 1-2b \\ 0 & -3b & 3 & b-1 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & b & 1 & 1 \\ 0 & 5b & -1 & 1-2b \\ 0 & 0 & 12 & -b-2 \end{array}\right)$$

$h(A)=h(A|b)=3$

$n=3$

$\dim V = 3-3=0$

$12z = -b-2$

$z = \frac{-b-2}{12}$

$5by + \frac{b+2}{12} = 1-2b$

$y = \frac{1}{5}\left[1-2b-\frac{b+2}{12}\right]$ $\triangle$

$-2x + b\cdot y + z = 1$

$x = -\frac{1-by-z}{2} = -\frac{1-\frac{1}{5}b\left[1-2b-\frac{b+2}{12}\right]+\frac{b+2}{12}}{2}$ $\circledast$

Solution: $\left[\circledast, \triangle, \frac{-b-2}{12}\right]$

V. $a \neq 1 \wedge a \neq 2 \wedge b \neq 0$

The solution can be find by Cramer rule

$$\det A_x = \begin{vmatrix} 1 & b & 1 \\ b & ab & 1 \\ 1 & b & a \end{vmatrix} = a^2b + b + b^2 - ab - b - ab^2 = a^2b + b^2 - ab - ab^2 =$$
$$= b\cdot(a^2 + b - a - ab)$$

$$\det A_y = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & a \end{vmatrix} = a^2b + 1 + 1 - b - a - a = \underline{\underline{a^2b + 2 - b - 2a}}$$

$$\det A_z = \begin{vmatrix} a & b & 1 \\ 1 & ab & b \\ 1 & b & 1 \end{vmatrix} = a^2b + b^2 + b - ab - ab^2 - b = \underline{\underline{a^2b + b^2 - ab - ab^2}}$$

Solution: $\left[\dfrac{\det A_x}{\det A}, \dfrac{\det A_y}{\det A}, \dfrac{\det A_z}{\det A} \right] =$

$$= \underline{\underline{\left[\frac{a^2 + b - a - ab}{(a-1)^2(a+2)}, \frac{a^2b + 2 - b - 2a}{b(a-1)^2(a+2)}, \frac{a^2 \cdot + b - a - ab}{(a-1)^2(a+2)} \right]}}$$

Try to solve b, 5a and 5b, 6a and 6b.