VI. QUADRATIC FORMS

Martina Bečvářová

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Faculty of Transportation Sciences, CTU in Prague

Definition

Let V be a vector space over $\mathbb{R}$, $M = \{v_1, v_2, \ldots, v_n\}$ be its basis, A be a symmetric matrix of size $n \times n$. By a *quadratic form on* V we mean a real function on V ($F : V \to \mathbb{R}$) defined as

$$F(x) = \langle x \rangle_M \cdot A \cdot \langle x \rangle_M^t \, , \, x \in V.$$

Note

1. The symbol $\langle x \rangle_M$ means the coordinates of x with respect to the basis M, so that $\langle x \rangle_M = (x_1, x_2, \ldots, x_n)$ for $x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$.
2. The quadratic form F can be written as

$$F(x) = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{12} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & a_{3n} & \ldots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$$

or

$$F(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = a_{11} x_1^2 + a_{22} x_2^2 + \cdots + a_{nn} x_n^2 + 2a_{12} x_1 x_2 + 2a_{13} x_1 x_3 + \cdots + 2a_{n-1,n} x_{n-1} x_n.$$

Theorem

If V is a vector space over $\mathbb{R}$ and F and G are two quadratic forms on V, then

a) $F + G$ is also a quadratic form on V,

b) αF is also a quadratic form on V for every $\alpha \in \mathbb{R}$.

Examples

1. Consider $F(x) = 2x^2 + 3y^2 + 2xy - 4xz + 6yz$. Decide whether F is a quadratic form on $\mathbb{R}^3$ (with respect to the standard basis). If yes, write its matrix.

Yes, it is a quadratic form (see the analytic expression in the definition) and its matrix is

$$A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 3 & 3 \\ -2 & 3 & 0 \end{pmatrix}.$$

2. Consider $F(x) = x^2 + y^2 + x + y - xz$. Decide whether F is a quadratic form on $\mathbb{R}^3$ (with respect to the standard basis). If yes, write its matrix.

No, it is not a quadratic form (see the analytic expression in the definition). The „problematic part“ is $x + y$.

Examples

1. Consider

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Decide whether the matrix A represents a quadratic form on $\mathbb{R}^3$ (with respect to the standard basis).

No, the matrix is not a square matrix.

2. Consider

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

Decide whether the matrix A represents a quadratic form on $\mathbb{R}^3$ (with respect to the standard basis).

No, the matrix is not a symmetric matrix.

3. Consider

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{pmatrix}.$$

Decide whether the matrix A represents a quadratic form on $\mathbb{R}^3$ (with respect to the standard basis). If yes, write its analytic expression.

The matrix is a symmetric matrix. Hence

$$F(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 - 3z^2 - 2xz + 4yz$$

is a quadratic form.

Definition

By the *rank of a quadratic form* we mean the rank of its matrix.

Note

1. If A is a regular matrix, then $F(x) = 0$, if and only if $x = o$.
2. If A is a non-regular matrix, then there exists $x \in V$, $x \neq o$, such that $F(x) = 0$.

Definition

The basis $N = \{u_1, u_2, \dots, u_n\}$ of a vector space V is called a *polar basis* of a quadratic form F, if its analytic expression is

$$F(x) = \begin{pmatrix} x_1 & x_2 & \dots & x_n \end{pmatrix} \cdot \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2.$$

Note

1. In this case, the matrix A is diagonal.
2. The form $F(x) = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$ is called a *polar expression of quadratic form* F.

Theorem

Let V be a vector space of dimension n over $\mathbb{R}$ and let $F : V \to \mathbb{R}$ be a quadratic form on V. Then there exists a polar basis $\{v_1, v_2, \dots, v_n\}$ of V of the quadratic form F that is, there exist $b_{11}, \dots, b_{nn} \in \mathbb{R}$ such that

$$F(x) = b_{11}x_1^2 + b_{22}x_2^2 + \cdots + b_{nn}x_n^2,$$

whenever $x = \sum_{i=1}^{n} x_i v_i$.

Definition

Let F be a quadratic form on a vector space V. Then F

a) is called *positive definite*, if $F(x) > 0$ for all non-zero x,
b) is called *positive semidefinite*, if $F(x) \geq 0$ for all x,
c) is called *negative definite*, if $F(x) < 0$ for all non-zero x,
d) is called *negative semidefinite*, if $F(x) \leq 0$ for all x,
e) is called *indefinite*, if $F(x) > 0$ for some x and $F(y) < 0$ for some y.

Theorem

Let $F(x) = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$ be a polar expression of a quadratic form on a vector space V. Then F is

a) *positive definite*, if and only if $a_{ii} > 0$ for all $i = 1, \ldots, n$,
b) *positive semidefinite*, if and only if $a_{ii} \geq 0$ for all $i = 1, \ldots, n$,
c) *negative definite*, if and only if $a_{ii} < 0$ for all $i = 1, \ldots, n$,
d) *negative semidefinite*, if and only if $a_{ii} \leq 0$ for all $i = 1, \ldots, n$,
e) *indefinite*, if and only if $a_{ii} > 0$ for some $i = 1, \ldots, n$ and $a_{jj} < 0$ for some $j = 1, \ldots, n$.

Definition

By a *signature* of a quadratic form we mean the 3-tuple of numbers which expresses the number p of positive entries a_{ii}, the number q of negative entries a_{ii} and the number r of zero entries a_{ii} in the polar expression of the quadratic form F. We write the signature as $\operatorname{sgn} F = (p, q, r)$.

Note

Let $F(x) = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2$ be a polar expression of a quadratic form on a vector space V. Then F is

a) *positive definite*, if and only if its signature is $\operatorname{sgn} F = (n, 0, 0)$,
b) *positive semidefinite*, if and only if its signature is $\operatorname{sgn} F = (k, 0, m)$, where $k, m \in \mathbb{N}$ and $k+m = n$,
c) *negative definite*, if and only if its signature is $\operatorname{sgn} F = (0, n, 0)$,
d) *negative semidefinite*, if and only if its signature is $\operatorname{sgn} F = (0, l, m)$, where $l, m \in \mathbb{N}$ and $l+m = n$,
e) *indefinite*, if and only, if its signature is $\operatorname{sgn} F = (k, l, m)$, where $k, l, m \in \mathbb{N}$ and $k + l + m = n$.

Theorem (the so called Sylvester's criterium)

The signature of a quadratic form is independent on the choice of a polar basis.

Methods how to find a polar expression of a quadratic form

Our objective now is to obtain a polar form for real symmentric matrices.

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I. Process of squares completing

Example

Consider the quadratic form $F : \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(x, y, z) = x^2 - 2xy + 4yz - 2y^2 + z^2.$$

Find the signature of the quadratic form.

By process of „completing the squares“ it is readily seen that

$$F(x,y,z) = x^2 - 2xy + 4yz - 2y^2 + z^2 = (x-y)^2 - y^2 + (z+2y)^2 - 4y^2 - 2y^2 =$$

$$= (x-y)^2 - 7y^2 + (z+2y)^2.$$

The expression in a polar form is

$$F(x',y',z') = (x')^2 - 7(y')^2 + (z')^2.$$

The polar form is of rank 3 and its singnature is $(2,1,0)$, hence a quadratic form is indefinite.

Alternatively, we can work with matrices. The matrix of F is

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

and the Jordan canonical matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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II. Symmetric elemetary matrix operations

A short theory

Definition

A matrix A is called *orthogonal* if A^{-1} exists and is equal to A^t. If there exists an orthogonal matrix U such that $B = U^tAU = U^{-1}AU$, then we say that B is *orthogonally similar to* A.

Theorem

If A is a square matrix over $\mathbb{R}$, then A is orthogonally similar to a diagonal matrix if and only if A is symmetric.

Example

Consider the quadratic form F on $\mathbb{R}^3$ given by

$$F(x,y,z) = x^2 + 2xy + 2xz + 2y^2 + 2z^2 + 2yz$$

expressed with respect to the canonical basis. Find a polar basis, polar expression and signature of the quadratic form.

In order to a diagonal matrix, we will introduce a special matrix. In its first part, there is a matrix of our quadratic form; in each row of the second part there are vectors from a basis of $\mathbb{R}^3$. In our example, we may chose the canonical (standard) basis, therefore $\{(1,0,0),(0,1,0),(0,0,1\}$. We will transform our matrix by symmetric elementary operations, thus we will make the same elementary operations to rows as well as to columns in the first part of our „special matrix".

The matrix of F is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

By process of „symmetric elemetary operations" it is readily seen that

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 \end{array}\right) \sim$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right).$$

In the rows of the second part of the last matrix, we can see the polar basis $B = \{(1,0,0),(1,-1,0),(1,0,-1)\}$. In the first part of the last matrix, we see the diagonal matrix which is the Jordan canonical matrix of the original matrix of the quadratic form considered, so that

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence the new analytic expression of our quadratic form with respect to B is

$$F(x',y',z') = (x')^2 + (y')^2 + (z')^2$$

and its signature is $(3,0,0)$. Our quadratic form is positive definite.

Note

The above described method is a suitable tool in linear algebra because it gives us a new analytic expression, a signature as well as a polar basis.

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III. Method of general upper minors

A short theory

We will use what we learnt in the chapter *Determinants*.

Theorem

Let A be the matrix of a quadratic form and let us denote by $\det A_1, \det A_2, \dots, \det A_n = \det A$ the sequence of general upper determinants of a matrix A. Then

1. if all entries of the sequence of general upper determinants are positive, then the quadratic form is positive definite.
2. if the sings of entries of the sequence of general upper determinants alternate, which means $-, +, -, +, \dots$, then the quadratic form is negative definite.

Examples

1. Consider the quadratic form $F : \mathbb{R}^3 \to \mathbb{R}$ given by the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Determine the signature of F.

We calculate the general upper determinants. We obtain the following sequence:

$$\det A_1 = |1| = 1, \quad \det A_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1, \quad \det A_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} = 1.$$

As we see, all entries of our sequence are positive, therefore our quadratic form is positive definite and its signature is $(3, 0, 0)$.

2. Consider the quadratic form $F : \mathbb{R}^3 \to \mathbb{R}$ given by the matrix

$$\begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Determine the signature of F.

We calculate the general upper determinants. We obtain the following sequence:

$$\det A_1 = |-2| = -2, \quad \det A_2 = \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1, \quad \det A_3 = \begin{vmatrix} -2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -3 \end{vmatrix} = -3.$$

As we see, the signs of the sequence of general upper determinants alternate (they are $-, +, -$), hence the quadratic form is negative definite and its signature is $(0, 3, 0)$.

3. Consider the quadratic form $F : \mathbb{R}^3 \to \mathbb{R}$ given by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}.$$

Determine the signature of F.

We calculate the general upper determinants. We obtain the following sequence

$$\det A_1 = |1| = 1, \quad \det A_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \det A_3 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -5.$$

As we see, the quadratic form is indefinite and its signature cannot be determined by using the method of general upper minors.

Note

The method described above is useful in mathematical analysis (determination of relative maxima and minima of functions $f : \mathbb{R}^n \to \mathbb{R}$; if the form is positive definite at x, then f has a relative minimum at x, and if the form is negative definite at x, then f has a relative maximum at x). We will study these applications in the lecture *Caculus I.*

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IV. Method of eigenvalues and eigenvectors

Theorem

Each real symmetric matrix is diagonalizable. All eigenvalues of a symmetric matrix are real. If u is an eigenvector associated with the eigenvalue λ_i and v is an eigenvector associated with the eigenvalue λ_j and $\lambda_i \neq \lambda_j$, then the vectors u and v are orthogonal, that is, $u \cdot v = 0$.

Note

It should observed that the polar basis contains eigenvectors, so that it is an orthogonal basis. Eigenvalues are on the diagonal of the Jordan canonical form.

Example

1. Consider the quadratic form $F : \mathbb{R}^3 \to \mathbb{R}$ given by the matrix

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}.$$

Find its polar basis and signature.

We calculate eigenvalues and eigenvectors:

$$\det(A-\lambda E)=\begin{pmatrix}-\lambda & 2 & 3\\ 2 & -\lambda & 0\\ 3 & 0 & -\lambda\end{pmatrix}=-\lambda(\lambda^2-13)=0.$$

As we see, the eigenvalues are

$$\lambda_1=0,\quad \lambda_2=\sqrt{13},\quad \lambda_3=-\sqrt{13}.$$

Hence the Jordan canonical form of the matrix A is

$$\begin{pmatrix}0 & 0 & 0\\ 0 & \sqrt{13} & 0\\ 0 & 0 & -\sqrt{13}\end{pmatrix},$$

its polar expression is $F(x',y',z')=0(x')^2+\sqrt{13}(y')^2-\sqrt{13}(z')^2$ and its signature is $(1,1,1)$.

Now, we can calculate three independent and orthogonal eigenvectors.

$$(A-0\cdot E)\cdot v_1^t=o^t=\begin{pmatrix}0 & 2 & 3\\ 2 & 0 & 0\\ 3 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\quad\Rightarrow\quad v_1=(0,3,-2),$$

$$(A-\sqrt{13}\cdot E)\cdot v_2^t=o^t=\begin{pmatrix}-\sqrt{13} & 2 & 3\\ 2 & -\sqrt{13} & 0\\ 3 & 0 & -\sqrt{13}\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\quad\Rightarrow\quad v_2=(\sqrt{13},2,3),$$

$$(A+\sqrt{13}\cdot E)\cdot v_3^t=o^t=\begin{pmatrix}\sqrt{13} & 2 & 3\\ 2 & \sqrt{13} & 0\\ 3 & 0 & \sqrt{13}\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\quad\Rightarrow\quad v_3=(-\sqrt{13},2,3).$$

The polar basis is $\{(0,3,-2),(\sqrt{13},2,3),(-\sqrt{13},2,3)\}$.

Exercises

1. Decide whether f is a quadratic form. If yes, write its matrix and calculate its signature.

a)
$$f(x_1,x_2,x_3)={x_1}^2+{x_2}^2+2{x_3}^2+4x_1x_2+2x_1x_3+2x_2x_3.$$

b)
$$f(x_1,x_2,x_3)={x_1}^2-2{x_2}^2+{x_3}^2+2x_1x_2+4x_1x_3+2x_2x_3.$$

c)
$$f(x_1,x_2,x_3)={x_1}^2-3{x_2}^2-2x_1x_2+2x_1x_3-6x_2x_3.$$

d)
$$f(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

e)
$$f(x_1, x_2, x_3, x_4) =$$
$${x_1}^2 + 2{x_2}^2 + {x_4}^2 + 4x_1x_2 + 4x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4.$$

f)
$$f(x_1, x_2, x_3) = {x_1}^2 + 5{x_2}^2 - 4{x_3}^2 + 2x_1x_2 - 4x_1x_3.$$

g)
$$f(x_1, x_2, x_3) = 4{x_1}^2 + {x_2}^2 + {x_3}^2 - 4x_1x_2 + 4x_1x_3 - 3x_2x_3.$$

h)
$$f(x_1, x_2, x_3) = 2x_1x_2 + x_1x_3 + x_2x_3.$$

i)
$$f(x_1, x_2, x_3) = 2{x_1}^2 + 18{x_2}^2 + 8{x_3}^2 - 12x_1x_2 + 8x_1x_3 - 27x_2x_3.$$

j)
$$f(x_1, x_2, x_3) = -12{x_1}^2 - 3{x_2}^2 + -12{x_3}^2 + 12x_1x_2 + -24x_1x_3 + 8x_2x_3.$$

2. Decide whether the matrices A and B represent quadratic forms . If yes, write their analytic expressions and express their signatures.

a)
$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

b)
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 12 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$$

c)
$$A = \begin{pmatrix} 0 & 2 & -1 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

d)
$$A = \begin{pmatrix} 10 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 3 & 1 \\ -3 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}.$$

e)
$$A = \begin{pmatrix} 10 & 1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 & 2 \\ 3 & 2 & 0 \end{pmatrix}.$$

f)
$$A=\begin{pmatrix}0&0&1\\0&1&2\\1&2&3\end{pmatrix},\qquad B=\begin{pmatrix}0&0&1\\3&0&2\\1&2&0\end{pmatrix}.$$

g)
$$A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix},\qquad B=\begin{pmatrix}0&0&1\\3&0&2\end{pmatrix}.$$

h)
$$A=\begin{pmatrix}1&0&-1\\0&0&1\\-1&1&2\end{pmatrix},\qquad B=\begin{pmatrix}0&0\\0&0\end{pmatrix}.$$

i)
$$A=\begin{pmatrix}2&0&-1\\0&1&0\\-1&0&1\end{pmatrix},\qquad B=\begin{pmatrix}1&2\\0&1\end{pmatrix}.$$

j)
$$A=\begin{pmatrix}11&1&-1\\1&8&1\\-1&1&1\end{pmatrix},\qquad B=\begin{pmatrix}1&2&2\\2&2&2\end{pmatrix}.$$

3. Calculate signatures of the quadratic forms f and g.

a)
$$f(x_1,x_2,x_3,x_4)=-4x_1x_4,$$
$$g(x_1,x_2)={x_1}^2-2x_1x_2+4{x_2}^2.$$

b)
$$f(x_1,x_2)={x_1}^2+26{x_2}^2+10x_1x_2,$$
$$g(x_1,x_2)={x_1}^2+56{x_2}^2+16x_1x_2.$$

c)
$$f(x_1,x_2,x_3)=8{x_1}^2-28{x_2}^2+14{x_3}^2+16x_1x_2+14x_1x_3+32x_2x_3,$$
$$g(x_1,x_2)={x_1}^2+4{x_2}^2+2{x_3}^2+2x_1x_2.$$

d)
$$f(x_1,x_2,x_3)=2{x_1}^2+3{x_2}^2-{x_3}^2+2x_1x_2+2x_2x_3,$$
$$g(x_1,x_2,x_3)=3{x_1}^2+2{x_2}^2+{x_3}^2-2x_1x_3.$$

e)
$$f(x_1,x_2)={x_1}^2+4x_1x_2-{x_2}^2,$$
$$g(x_1,x_2)={x_1}^2+6x_1x_2+5{x_2}^2.$$

f)
$$f(x_1, x_2, x_3) = 21{x_1}^2 - 18{x_2}^2 + 6{x_3}^2 + 4x_1x_2 + 28x_1x_3 + 6x_2x_3,$$
$$g(x_1, x_2, x_3) = 11{x_1}^2 + 6{x_2}^2 + 6{x_3}^2 - 12x_1x_2 + 12x_1x_3 - 6x_2x_3.$$

g)
$$f(x_1, x_2, x_3) = 14{x_1}^2 - 4{x_2}^2 + 17{x_3}^2 + 8x_1x_2 - 40x_1x_3 - 26x_2x_3,$$
$$g(x_1, x_2, x_3) = 9{x_1}^2 + 6{x_2}^2 + 6{x_3}^2 + 12x_1x_2 - 10x_1x_3 - 2x_2x_3.$$

h)
$$f(x_1, x_2, x_3) = 3{x_2}^2 + 3{x_3}^2 + 4x_1x_2 + 4x_1x_3 - 2x_2x_3,$$
$$g(x_1, x_2, x_3) = 7{x_1}^2 + 7{x_2}^2 + 7{x_3}^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.$$

i)
$$f(x_1, x_2, x_3) = {x_1}^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3,$$
$$g(x_1, x_2, x_3) = 3{x_1}^2 + 3{x_2}^2 - {x_3}^2 - 6x_1x_3 + 4x_2x_3.$$

j)
$$f(x_1, x_2, x_3) = 11{x_1}^2 + 5{x_2}^2 + 2{x_3}^2 + 16x_1x_2 + 4x_1x_3 - 20x_2x_3,$$
$$g(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 + 5{x_3}^2 - 6x_1x_2 - 2x_1x_3 + 2x_2x_3.$$