I. VECTOR SPACES

Martina Bečvářová

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Faculty of Transportation Sciences, CTU in Prague

Definition

By a *vector space* we shall mean a set V on which there are defined two operations, one called addition and the other called multipication by scalars, such that the following properties hold:

1. $x + y = y + x$ for all $x, y \in V$;
2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in V$;
3. there exists an element $o \in V$ such that $x + o = x$ for all $x \in V$;
4. for every $x \in V$ there exists an element $-x \in V$ such that $x + (-x) = o$;
5. $\lambda(x + y) = \lambda x + \lambda y$ for all $x, y \in V$ and all scalars λ;
6. $(\lambda + \mu)x = \lambda x + \mu x$ for all $x \in V$ and all scalars λ, μ;
7. $(\lambda\mu)x = \lambda(\mu x)$ for all $x \in V$ and all scalars λ, μ;
8. $1 \cdot x = x$ for all $x \in V$.

Notes

1. When the scalars are real numbers, we shall often speak of a *real vector space*; and when the scalars are complex numbers we shall talk of a *complex vector space*.
2. The first axiom, so that $x + y = y + x$, says that the operation $+$ is commutative.
3. The second axiom, so that $(x + y) + z = x + (y + z)$, says that the operation $+$ is associative.
4. Elements of V are called *vectors*.
5. The vector o is called a *zero vector*. For example, in the vector space $\mathbb{R}^3$ a zero vector is $o = (0, 0, 0)$.
6. Vector $-x$ is called an *opposite vector* to the vector x. For example, in the vector space $\mathbb{R}^4$, $x = (1, -2, 0, 6)$, $-x = (-1, 2, 0, -6)$.

Attention

We use the same symbol $+$ for addition of vectors and for addition of real (complex) numbers.

We use the same symbol $\cdot$ for multiplication of a vector by scalar and for multiplication scalar by scalar.

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Some examples of vector spaces

1. The set $\mathbb{R}^n$ of n-tuples $(x_1, \ldots, x_n)$ of real numbers is a real vector space under the following component-wise definitions of addition and multiplication by scalars:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$
$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n).$$

• Geometrically, $\mathbb{R}^2$ represents the cartesian plane, and $\mathbb{R}^3$ represents the three-dimensional cartesian space.

• Similarly, the set $\mathbb{C}^n$ of n-tuples of complex numbers can be considered as a real vector space (real scalars) as well as a complex vector space (complex scalars).

2. Let $\mathbb{R}_{n+1}[x]$ be the set of polynomials of degree at most n with real coefficients. Define the addition by setting

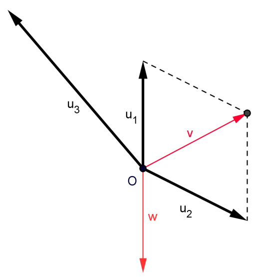
$$(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n) =$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n,$$

and a multiplication by scalars by setting

$$\lambda(a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n) = \lambda a_0 + \lambda a_1 x + \lambda a_2 x^2 + \cdots + \lambda a_n x^n.$$

Then $\mathbb{R}_{n+1}[x]$ is a real vector space.

3. Geometrical model



$$\overrightarrow{v} = \overrightarrow{u_1} + \overrightarrow{u_2}, \qquad \overrightarrow{w} = -\overrightarrow{u_1}, \qquad \overrightarrow{o}$$

Comments

It should be noted that in the definition of a vector space the scalars need not be restricted to be real or complex numbers. They can in fact belong to any „field“ F. Although in what follows we shall find it convenient to say that „V is a vector space over a field F“ to indicate that the scalars come from a field F, we shall in fact assume throughout that F is either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

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We now list some basic properties of multiplication by scalars in a vector space that follow from the above eight axioms.

Theorem

If V is a vector space over $\mathbb{R}$ (or $\mathbb{C}$), then

1. $\lambda \cdot o = o$ for every $\lambda \in \mathbb{R}$ (or $\mathbb{C}$);
2. $0 \cdot x = o$ for every $x \in V$;
3. if $\lambda x = o$ then either $\lambda = 0$ or $x = o$;
4. $(-\lambda)x = -(\lambda x) = \lambda(-x)$ for all $x \in V$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$);
5. $-(-x) = x$ for every $x \in V$;
6. $\lambda(x - y) = \lambda x - \lambda y$ for all $x, y \in V$ and $\lambda \in \mathbb{R}$ (or $\mathbb{C}$);
7. $(\lambda - \mu)x = \lambda x - \mu x$ for all $x \in V$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$).

Using the axioms try to proved (or verify) these statements.

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In order to study vector spaces we firstly introduce subspaces, thus those subsets that are also vector spaces.

Definition

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). By a *subspace* of V we mean a non-empty subset W of V that is closed under the operations of V, that is,

(1) if $x, y \in W$, then $x + y \in W$;

(2) if $x \in W$ and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$), then $\lambda x \in W$.

Note that (1) says that sums of elements of W belong to W, and (2) says that scalar multiples of elements of W belong to W.

Notes

1. Every vector space V is (trivially) a subspace of itself; this is the largest subspace of V.
2. Obviously, $V = \{o\}$ is the smallest subspace of V.
3. These two subspaces are called *trivial subspaces.*
4. Other subspaces (if exist) are called *non-trivial subspaces.*

Some examples of subspaces

1. In the plane $\mathbb{R}^2$ every line through the origin can be expressed in the form

$$L = \{(x, y); \alpha x + \beta y = 0\};$$

the slope of the line is given by $\tan \phi = -\frac{\alpha}{\beta}$, if $\beta \neq 0$. Every line through the origin is a subspace of $\mathbb{R}^2$. Verify it!

2. In the space $\mathbb{R}^3$ every plane through the origin can be expressed in the form

$$P = \{(x, y, z); \alpha x + \beta y + \gamma z = 0\}.$$

Every plane through the origin is a subspace of $\mathbb{R}^3$. Verify it!

3. The set of real functions on the interval (a,b) with usual operations $f + g$, λf (i.e. for every $x \in (a, b)$ we define $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x)$) is a vector space. Consider now the subset of all continuous functions with the same operations. It is a subspace of the vector space. Verify it!

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Notes on the intersection of subspaces

Suppose now that A and B are subspaces of a vector space V over $\mathbb{R}$ (or $\mathbb{C}$) and consider their *intersection* $A \cap B$.

(1) We know that the zero vector o is contained in both A and B, so that $o \in A \cap B$ so that $A \cap B$ is not empty.

(2) Now, if $x, y \in A \cap B$, then $x, y \in A$ gives $x + y \in A$, and $x, y \in B$ gives $x + y \in B$, whence we have $x + y \in A \cap B$.

(3) Likewise, if $x \in A \cap B$, then $x \in A$ gives $\lambda x \in A$, and $x \in B$ gives $\lambda x \in B$ for every real (complex) number λ, whence we have $\lambda x \in A \cap B$.

Thus we see that $A \cap B$ is a subspace of V.

Theorem

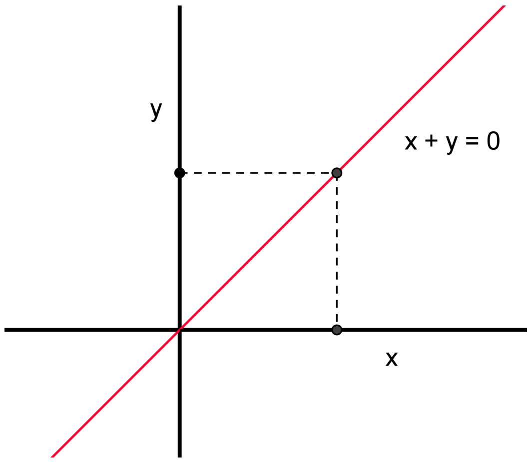
The intersection of any collection of subspaces of a vector space V is a subspace of V.

Notes

In contrast with this the above situation, we note that the *union of a collection of subspaces of a vector space* V is not in general a subspace of V.

Example

In $\mathbb{R}^2$ the x-axis X and the y-axis Y are subspaces, but their union is not. For example, we have a vector $(1,0) \in X$ and we have a vector $(0,1) \in Y$ but a vector $(1,0)+(0,1) = (1,1) \notin X \cup Y$. So the subset $X \cup Y$ is not closed under addition and therefore cannot be a subspace.



x-axis $= X$, $\quad$ y-axis $= Y$, $\quad \overrightarrow{u} = \overrightarrow{x} + \overrightarrow{y}$, hence $x + y = 0$.

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Definition

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$) and let S be a non-empty subset of V. We say that $v \in V$ is a *linear combination of elements of* S if there exist $x_1, \ldots, x_n \in S$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (or $\mathbb{C}$) such that

$$v = \lambda_1 x_1 + \cdots + \lambda_n x_n = \sum_{j=1}^{n} \lambda_j x_j.$$

Notes

1. When $j \geq 1$ and $\lambda_j = 0$ for every j a linear combination is called a *trivial linear combination.*
2. In other cases, a linear combination is called a *non-trivial combination.*

Examples

1. Let $x_1 = (1,2,4,-1)$, $x_2 = (0,-1,2,3)$, $x_3 = (0,0,1,-1)$ and $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 0$. Calculate $v = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$:

$$v = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = -1 \cdot (1,2,4,-1) + 2 \cdot (0,-1,2,3) + 0 \cdot (0,0,1,-1) =$$
$$= (-1,-2,-4,1) + (0,-2,4,6) + (0,0,0,0) = (-1,-4,0,7).$$

2. Let $x_1 = (1, 2)$, $x_2 = (0, 2, 3)$ and $\lambda_1 = 12$, $\lambda_2 = -3$. Calculate $v = \lambda_1 x_1 + \lambda_2 x_2$.

The sum is not defined because $x_1 \in \mathbb{R}^2$ and $x_2 \in \mathbb{R}^3$.

Notes

It is clear that if $v = \lambda_1 x_1 + \cdots + \lambda_n x_n$ and $w = \mu_1 x_1 + \cdots + \mu_n x_n$ are linear combinations of elements of S, then so are $v + w$ and λv for every $\lambda \in \mathbb{R}$ (or $\mathbb{C}$).

Thus the set of linear combinations of elements of a finite set S is a subspace of V. We call it the *subspace spanned by* S and denote it by Span S or $< S >$.

Definition

If Span S is the whole of V, we often say that S is a *spanning set for* V.

Some examples:

1. Consider the subset $S = \{(1, 0), (0, 1)\}$ of the cartesian plane $\mathbb{R}^2$. For every $(x, y) \in \mathbb{R}^2$ we have

$$(x, y) = (x, 0) + (0, y) = x(1, 0) + y(0, 1),$$

so that every element of $\mathbb{R}^2$ is a linear combination of elements of S. Thus S spans $\mathbb{R}^2$.

2. Consider the subset $S = \{(1, 0, 0), (0, 0, 1)\}$. $\operatorname{Span} S$ is

$$\operatorname{Span} S\{(1, 0, 0), (0, 0, 1)\} = \{a(1, 0, 0) + b(0, 0, 1); a, b \in \mathbb{R}\} = \{(a, 0, b); a, b \in \mathbb{R}\}.$$

In other words, the subspace of $\mathbb{R}^3$ that is spanned by the subset $S = \{(1, 0, 0), (0, 0, 1)\}$ is the „x, z-plane“.

3. More generally, if $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is an n-tuple with the 1 in the j-th position, then we have

$$(x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n,$$

and so $\{e_1, \ldots, e_n\}$ spans $\mathbb{R}^n$.

4. Consider the subset $S = \{(2, 1), (1, 1), (-1, 1)\}$ of the plane $\mathbb{R}^2$. For every $(x, y) \in \mathbb{R}^2$ we have

$$(x, y) = a(2, 1) + b(1, 1) + c(-1, 1), \quad \text{where} a, b, c \in \mathbb{R},$$

so that every element of $\mathbb{R}^2$ is a linear combination of elements of S Thus S spans $\mathbb{R}^2$.

Indeed, for every $(x, y) \in \mathbb{R}^2$ we can find a, b, c by solving the following system of linear equations

$$\begin{aligned} 2a + b - c &= x, \\ a + b + c &= y. \end{aligned}$$

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Definition

A non-empty subset S of a vetor space V over $\mathbb{R}$ (or $\mathbb{C}$) is said to be *linearly independent*, if the only way of expressing the vector o as a linear combination of elements of S is the trivial one; equivalently, if $x_1, \ldots, x_n \in S$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (or $\mathbb{C}$), then

$$\lambda_1 x_1 + \ldots \lambda_n x_n = o \quad \Rightarrow \quad \lambda_1 = \cdots = \lambda_n = 0.$$

A subset that is not linearly independent is said to be *linearly dependent*.

Notes

1. For a linearly dependent subset S it is evident that the vector o can be expressed as a non-trivial combination of elements of S, thus if $x_1, \ldots, x_n \in S$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (or $\mathbb{C}$), then

$$\lambda_1 x_1 + \ldots \lambda_n x_n = o \quad \text{with at least one} \quad \lambda_j \neq 0.$$

2. No linearly independent subset of a vector space V can contain the vector o.

Examples

1. Let $S = \{(0,2,0), (3,0,0), (0,0,-5)\}$. The subset S is linearly independent. Verify it!

Let us denote $x = (0,2,0)$, $y = (3,0,0)$, $z = (0,0,-5)$. We write a linear combination of these three vectors and equal it to the zero vector. We obtain

$$\begin{aligned} ax + by + cz &= o, \\ a(0,2,0) + b(3,0,0) + c(0,0,-5) &= (0,0,0), \\ (3b, 2a, -5c) &= (0,0,0). \end{aligned}$$

We obtain the following system of linear equations

$$\begin{aligned} 3b &= 0, \\ 2a &= 0, \\ -5c &= 0. \end{aligned}$$

We can see that $a = 0$, $b = 0$, $c = 0$, hence only a trivial linear combination of vectors x, y and z gives the zero vector, so that the vectors x, y and z are linearly independent.

2. Let $S = \{(1,2,1), (1,3,2), (1,1,0)\}$. The subset S is linearly dependent. Verify it!

Let us denote $x = (1,2,1)$, $y = (1,3,2)$, $z = (1,1,0)$. We write a linear combination of these three vectors and equal it to the zero vector. We obtain

$$\begin{aligned} ax+by+cz &= o, \\ a(1,2,1)+b(1,3,2)+c(1,1,0) &= (0,0,0), \\ (a+b+c, 2a+3b+c, a+2b) &= (0,0,0). \end{aligned}$$

We obtain the following system of linear equations

$$\begin{aligned} a+b+c &= 0, \\ 2a+3b+c &= 0, \\ a+2b &= 0. \end{aligned}$$

Multiply the second equation by -1 and add the first equation. We obtain the following system of two identical equations

$$\begin{aligned} a+2b &= 0, \\ a+2b &= 0. \end{aligned}$$

We see that $a=-2b$, where $b \in \mathbb{R}$. We can easily calculate that $c=b$, hence the solution of our system is $(-2b,b,b)=b(-2,1,1)$ for every real number b. We can go back and we see that

$$-2bx+by+bz = b\cdot(-2x+y+z) = o, \quad \text{for every}\, b \in \mathbb{R}.$$

In particular, $-2x+y+z=o$, thus the zero vector o is expressed as a non-trivial linear combination of vectors x, y and z, so that the vectors are linearly dependent.

A faster method for solving our problem

1. Let $S=\{(0,2,0),(3,0,0),(0,0,-5)\}$. The subset S is linearly independent.

$$\begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

We write the vectors in the rows (the first one in the first row, the second one in the second row and the third one in the third row) and we obtain *a matrix.* Now we interchange the first row and the second row. We obtain the matrix which has below the diagonal only zeros. We have three non-zero rows in the so-called a *row-echelon* (or *stairstep*) matrix, hence vectors are linearly independent.

2. Let $S=\{(1,2,1),(1,3,2),(1,1,0)\}$. The subset S is linearly dependent.

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We write the vectors in the rows (the first one in the first row, the second one in the second row and the third one in the third row) and we obtain a *matrix.* Now we multiply the first row by -1 and add the second row, we multiply the first row by -1 and add the third row. We obtain a matrix which has two identical rows. We see that our vectors are linearly dependent. We can also continue. Now we multiply the second row by -1 and add the third row. We obtain one row with zeros only, so that we have the zero vector in the last row, hence our vectors are linearly dependent.

We can use these three types of matrices modifications:

1. We can replace one row by another row (we can interchange two rows).
2. We can multiply one row by a non-zero real (complex) number.
3. We can add two or more rows together.

These modifications do not change the independence (dependence) of vectors!

Examples

1. Let $S = \{(1,2,1,1),(1,3,2,2),(1,1,1,1,)\}$. Is the subset S linearly dependent? Justify your answer!

$$\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The vectors are linearly independent.

2. Let $S = \{(-1,-2,-1),(1,0,1),(1,1,1),(3,3,3)\}$. Is the subset S linearly dependent? Justify your answer!

The vectors are linearly dependent because when we multiply vector $(1,1,1)$ by 3 we obtain vector $(3,3,3)$. Another argument: we have four vectors from R^3 and four vectors in this space are always linearly dependent. Or we can calculate as follows

$$\begin{pmatrix} -1 & -2 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & -1 \\ 0 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & -2 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). If S is a subset of V containing at least two elements, then the following statements are equivalent:

(1) S is linearly dependent;

(2) at least one element of S can be expressed as a linear combination of other elements S.

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Now we combine the notions of a linearly independent set and a spanning set to obtain the following important notion.

Definition

By a *basis* of a vector space V we mean a linearly independent subset of V that spans V.

Examples showing differences between basis and span

1. The set $\{(1,0),(0,1)\}$ is a basis of the plane $\mathbb{R}^2$. It is evident.

2. The set $\{(1,1),(1,-1)\}$ is a basis of the plane $\mathbb{R}^2$. For every $(x,y) \in \mathbb{R}^2$ we have $(x,y) = \lambda(1,1) + \mu(1,-1)$ where $\lambda = \frac{1}{2}(x+y)$ and $\mu = \frac{1}{2}(x-y)$. So $\{(1,1),(1,-1)\}$ spans $\mathbb{R}^2$. And if $\alpha(1,1)+\beta(1,-1) = (0,0)$ then $\alpha+\beta = 0$ and $\alpha-\beta = 0$, whence $\alpha = \beta = 0$, so $\{(1,1),(1,-1)\}$ is also linearly independent.

3. Let $\{(1,1),(1,-1),(0,1)\}$. It is not a basis of the plane $\mathbb{R}^2$. The vectors $(1,1)$, $(1,-1)$, $(0,1)$ are linearly dependent. The set $\{(1,1),(1,-1),(0,1)\}$ spans the plane $\mathbb{R}^2$.

4. $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$. It is evident.

5. Let $\{(1,0,0),(0,1,0)\}$. It is not a basis of $\mathbb{R}^3$. Every $(x,y,z) \in \mathbb{R}^3$ cannot be written as a linear combination of $(1,0,0)$ and $(0,1,0)$.

6. More generally, $\{e_1,e_2,\ldots,e_n\}$ is a basis of $\mathbb{R}^n$. This basis is called a *canonical basis* or *natural basis.*

7. Clearly, $\{1,x,x^2,\ldots,x^n\}$ is a basis of the space $\mathbb{R}_{n+1}[x]$. It is evident.

Theorem

If V has a finite basis B, then every basis of V has the same number of elements as B.

Definition

By a finite-dimensional vector space we shall mean a vector space V that has a finite basis. The number of elements in any basis of V is called the *dimension* of V and will be denoted by $\dim V$.

Examples

1. The space $\mathbb{R}^n$ has dimension n.

2. The vector space of complex numbers over real numbers has dimension 2 (one of its basis is for example $\{1,i\}$).

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A fundamental characterization of bases is the following.

Theorems

• A non-empty subset S of a vector space V is a basis, if and only if every element of V can be expressed in a unique way as a linear combination of elements of S.

• Let V be a vector space that is spanned by the set $\{v_1, v_2, \dots, v_n\}$. If $I = \{w_1, w_2, \dots, w_m\}$ is a linearly independent subset of V, then necessarily $m \leq n$.

• Every linearly independent subset I of a finite-dimensional vector space can be enlarged to a basis.

• If V is of dimension n, then every linearly independent set consisting of n elements is a basis of V.

• If S is a subset of V, then the following statements are equivalent:

(1) S is a basis;

(2) S is a maximal independent subset;

(3) S is a minimal spannig set.

• If V is of dimension n, then every subset containing more than n elemetns is linearly dependent; moreover, no subset containing less than n elements can span V.

We now enquire about bases for subspaces of given finite-dimensional vector space.

Theorem

Let V be a finite-dimensional vector space. If W is a subspace of V, then W is also of finite dimension, and $\dim W \leq \dim V$. Moreover, we have $\dim W = \dim V$, if and only if $W = V$.

Theorem

Let V be a finite-dimensional vector space, U and W be subspaces of V. Then

$$\dim (U + W) + \dim (U \cap W) = \dim U + \dim W,$$

where $U + W$ means $\langle U \cup W \rangle$.

Examples

1. Choose a basis from the set S spanning a vector space V:
$S = \{(1, 0, 2, -3), (3, 2, 1, -5), (-1, 2, 1, -2), (-3, 0, 2, 0)\}$.

We consider the vectors as rows and we transform the matrix obtained into the a *row-echelon* matrix.

$$\begin{pmatrix} 1 & 0 & 2 & -3 \\ 3 & 2 & 1 & -5 \\ -1 & 2 & 1 & -2 \\ -3 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 2 & -5 & 4 \\ 0 & 2 & 3 & -5 \\ 0 & 2 & 3 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 2 & -5 & 1 \\ 0 & 0 & 8 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see three linearly independent vectors in three rows, so that three linearly independent vectors which span our vector space. One basis of our vector space is $\{(1,0,2,-3),(0,2,-5,1),(0,0,8,-6)\}$. The dimension of vector space V is 3.

2. Choose a basis from the set S spanning a vector space V:
$S=\{(1,1,2),(3,0,3),(2,2,2),(-3,0,0)\}$.

$$\begin{pmatrix} 1 & 1 & 2\\ 3 & 0 & 3\\ 2 & 2 & 2\\ -3 & 0 & 0\end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2\\ 0 & -3 & -3\\ 0 & 0 & -2\\ 0 & 1 & 2\end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2\\ 0 & -3 & -3\\ 0 & 0 & -2\\ 0 & 0 & 3\end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2\\ 0 & -3 & -3\\ 0 & 0 & -2\\ 0 & 0 & 0\end{pmatrix}$$

We see three linearly independent vectors in three rows, thus three linearly independent vectors which span our vector space. One basis of our vector space is $\{(1,1,2),(0,-3,-3),(0,0,-2)\}$. The dimension of vector space V is 3.

3. From the set S spanning a vector space V choose a basis which contains the vector $(2,3,5,4)$. $S=\{(1,2,5,3),(1,1,0,1),(1,1,1,1)\}$.

$$\left(\begin{array}{cccc} 2 & 3 & 5 & 4\\ \hdashline 1 & 2 & 5 & 3\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1\end{array}\right) \sim \begin{pmatrix} 2 & 3 & 5 & 4\\ 0 & -1 & -5 & -2\\ 0 & 1 & 5 & 2\\ 0 & 1 & 3 & 2\end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 5 & 4\\ 0 & -1 & -5 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & -2 & 0\end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 5 & 4\\ 0 & -1 & -5 & -2\\ 0 & 0 & -2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

We see that vectors $(2,3,5,4)$, $(1,2,5,3)$, $(1,1,0,1)$, $(1,1,1,1)$ are linearly dependent, thus the vector $(2,3,5,4)$ are elements of $\langle S\rangle$. We see three linearly independent vectors in the first three rows of the last matrix, so that our basis has three members. One basis of our vector space is $\{(2,3,5,4),(0,-1,-5,-2),(0,0,-2,0)\}$. The dimension of vector space V is 3.

4. From the set S spanning a vector space V choose a basis which contains a vector $(1,1,0,0)$. $S=\{(1,0,1,1),(2,1,0,1)\}$.

$$\left(\begin{array}{cccc} 1 & 1 & 0 & 0\\ \hdashline 1 & 0 & 1 & 1\\ 2 & 1 & 0 & 1\end{array}\right) \sim \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & -1 & 1 & 1\\ 0 & -1 & 0 & 1\end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & -1 & 1 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}$$

We see that vectors $(1,1,0,0)$, $(1,0,1,1)$, $(2,1,0,1)$ are linearly independent, therefore the vector $(1,1,0,0)$ cannot be expressed as a linear combination of vectors $(1,0,1,1)$, $(2,1,0,1)$ so our example has no solution.

* * * *

Definition

Let V be a vector space, $B = \{v_1, v_2, \dots, v_n\}$ be a basis of V, $v \in V$. By *coordinates* of a vector v with respect to the basis B we understand an ordered n-tuple of real numbers (or complex numbers) $(a_1, a_2, \dots, a_n)$ such that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = \sum_{j=1}^{n} a_j v_j.$$

We write $\langle v \rangle_B = (a_1, a_2, \dots, a_n)$.

Notes

1. For every vector its coordinates are uniquely determined n-tuple.
2. A modification of a basis means a modification of coordinates.

Examples

1. Let V be a vector space, let $B = \{(1,2,3), (0,1,1)\}$ be a basis of V and let $x = (2,3,5)$, $y = (2,3,0)$. Calculate $\langle x \rangle_B$ and $\langle y \rangle_B$.

Let us denote $v_1 = (1,2,3)$ and $v_2 = (0,1,1)$. According to the definition of coordinates we can write

$$\begin{aligned} av_1 + bv_2 &= x, \\ a(1,2,3) + b(0,1,1) &= (2,3,5), \\ (a, 2a+b, 3a+b) &= (2,3,5). \end{aligned}$$

We obtain the following system of three linear equations

$$\begin{aligned} a &= 2, \\ 2a + b &= 3, \\ 3a + b &= 5. \end{aligned}$$

From the first equation we see that $a = 2$, from the second equation we obtain $b = 3 - 2a = 3 - 4 = -1$. Now we must verify that our solution $(2, -1)$ satisfies also the third equation. We see that on the left hand side of the equation is $3a + b = 3 \cdot 2 - 1 = 5$, on the right hand side of the equation is 5. $5 = 5$. Our system of equations has a solution $a = 2$, $b = -1$, so the vector x has coordinates $(2, -1)$ and we have $\langle x \rangle_B = (2, -1)$.

Similarly, for the vector $y = (2,3,0)$,

$$\begin{aligned} av_1 + bv_2 &= y, \\ a(1,2,3) + b(0,1,1) &= (2,3,0), \\ (a, 2a+b, 3a+b) &= (2,3,0). \end{aligned}$$

We obtain the following system of three linear equations

$$\begin{aligned} a &= 2, \\ 2a + b &= 3, \\ 3a + b &= 0. \end{aligned}$$

From the first equation we see that $a = 2$, from the second equation we obtain $b = 3-2a = 3-4 = -1$. Now we must verify that our solution $(2, -1)$ is also a solution of the third equation. We see that on the left hand side of the equation is $3a + b = 3 \cdot 2 - 1 = 5$, on the right hand side of the equation is 0. Our system of equations has no solution, so the vector y cannot be expressed as a linear combination of vectors v_1 and v_2, therefore $y \notin V$.

$*\quad*\quad*\quad*$

Exercises

1. Decide whether the vectors are linearly dependent.

a) $v = (2, 1, 3, 1), u = (1, 2, 0, 1), w = (-1, 1, -3, 0)$.
b) $v = (2, 3, -5), u = (1, -1, 1), w = (3, 2, -2)$.
c) $v = (1, 0, 3), u = (-3, 0, -9), w = (1, 1, 2)$.
d) $v = (3, 4, 3), u = (1, 3, -1), w = (1, -1, 1)$.
e) $v = (1, 2, 0, 0), u = (0, 1, 1, 0), w = (1, 0, 0, 1), q = (1, 1, -1, 1)$.
f) $v = (4, 7, 1, 0), u = (2, 3, -1, 2), w = (1, 2, 1, -1), q = (5, 7, -4, 7)$.
g) $v = (2, 1, 1, 1), u = (1, 2, -1, 2), w = (1, -1, -1, 1), q = (1, 2, 2, -2)$.
h) $p = (2, 1, 3, -1), q = (-1, 1, -3, 1), r = (4, 5, 3, -1), s = (1, 5, -3, 1)$.
ch) $p = (1, 2, 1), q = (0, 1, -1), r = (2, -1, 1), s = (1, 0, 1)$.

2. Let $a \in \mathbb{R}$ be a parameter. Decide when the vectors are linearly dependent and when independent.

a) $v = (a, -2, 1), u = (3, 2a, -1), w = (a^2, 1, a - 1)$.
b) $v = (a, -4, -1), u = (4, -6, -3), w = (1, 1, -a)$.
c) $v = (2, 3, -1), u = (a, 4, 2)$.
d) $v = (1, 1, 1), u = (1, a, 1), w = (2, 2, a)$.
e) $u = (a, -4, -1), v = (4, -6, -3), w = (1, 1, -a)$

3. Find a basis of the vector space V spanned by next vectors. Calculate its dimension.

a) $u_1 = (1, 2, 3, 4), u_2 = (1, 5, 1, 2), u_3 = (1, 1, 2, 3)$.
b) $u_1 = (1, 2, 3, 2), u_2 = (0, 1, 1, 0), u_3 = (1, 0, 1, 2)$.
c) $u_1 = (1, 2, 3, 2), u_2 = (0, 1, -1, 3), u_3 = (1, 2, 1, 6), u_4 = (2, 4, 3, 5)$.
d) $u_1 = (3, 1, 5, 4), u_2 = (2, 2, 3, 3), u_3 = (1, -1, 2, 1), u_4 = (1, 3, 1, 2)$.
e) $u_1 = (1, 0, 2, -3), u_2 = (3, 2, 1, -5), u_3 = (-1, 2, 1, -2), u_4 = (-3, 0, 2, 0)$.
f) $u_1 = (3, 1, 5, 4), u_2 = (2, 2, 3, 3), u_3 = (1, -1, 2, 1), u_4 = (1, 3, 1, 2)$.
g) $u_1 = (3, -1, -3, 2), u_2 = (1, 2, 0, -3), u_3 = (1, 2, 1, 2), u_4 = (5, 1, -3, 2)$.
f) $u_1 = (5, 7, -1, 3), u_2 = (1, -3, 8, 2), u_3 = (9, 17, -10, 4), u_4 = (-2, 6, -16, -4)$.

4. Find a basis of the vector space spanned by the set $S = \{u_1, u_2, u_3\}$ which contains the vector v.

a) $u_1 = (0, 1, -3, 4), u_2 = (2, 2, 2, 2), u_3 = (1, -1, 3, 7)$ and $v = (1, 4, -4, -1)$.

b) $u_1 = (1, 2, 2, -2), u_2 = (1, 2, -1, -2), u_3 = (2, 1, 1, 1)$ and $v = (1, -1, -1, -1)$.

c) $u_1 = (1, 3, 5), u_2 = (3, 9, 15), u_3 = (1, 0, 2)$ and $v = (8, 25, 40)$.

5. Calculate $\langle u \rangle_M$ or $\langle v \rangle_M$

a) $u = (-10, 7, -4),\ M = \{(2, 1, 3), (-3, 1, -2), (5, -2, 4)\}$.

b) $u = (29, 12, 5),\ M = \{(-3, 2, -4), (5, -3, 2), (0, 6, -3)\}$.

c) $u = (-5, 17, -11),\ M = \{(1, 2, 1), (3, -2, 7), (11, -2, 23)\}$.

d) $u = (10, 1, 9, 33),\ M = \{(4, 0, -6, 13), (2, 3, -4, 7), (1, -2, 5, 3), (3, 0, 14, 10)\}$.

e) $u = (9, -26, 0, 21),\ M = \{(4, -6, 1, 9), (2, -3, -2, 5), (1, 2, 1, 2), (-1, 6, -2, -2)\}$.

f) $u = (-1, -2, 3, 5),\ M = \{(-1, 0, 2, 4), (2, 3, -5, 1), (1, 2, -3, 2), (2, 1, -4, 2)\}$.

g) $u = (-1, 0, 16, 15),\ M = \{(1, 2, -1, 3), (2, 1, 0, 0), (0, -1, 3, 0)\}$.

h) $u = (0, 0, 1, 5),\ M = \{(1, 2, 3, 4), (4, 3, 2, 1), (3, 1, 0, 2)\}$.

i) $u = (2, 4, 3, -2),\ M = \{(1, 0, 0, 1), (0, 1, 0, 1), (0, 1, 1, 0)\}$.

j) $u = (2, -2, 2),\ v = (7, -7, -2),\ M = \{(5, 2, 3), (4, 3, 2)\}$.

k) $u = (3, -1, 7),\ v = (-1, 3, -7),\ M = \{(1, 1, -1), (2, 0, 3)\}$.

l) $u = (7, 9, 2),\ v = 3(0, 1, 2) + 4(1, 0, 2) + 5(1, 2, 0),\ M = \{(0, 1, 2), (1, 0, 2), (1, 2, 0)\}$.