So far we have described plane curves by giving $y$ as a function of $x$ [$y = f(x)$] or $x$ as a function of $y$ [$x = g(y)$] or by giving a relation between $x$ and $y$ that defines $y$ implicitly as a function of $x$ [$f(x, y) = 0$]. In this chapter we discuss two new methods for describing curves.

Some curves, such as the cycloid, are best handled when both $x$ and $y$ are given in terms of a third variable $t$ called a parameter [$x = f(t), y = g(t)$]. Other curves, such as the cardioid, have their most convenient description when we use a new coordinate system, called the polar coordinate system.
Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form \( y = f(x) \) because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time and so we can write \( x = f(t) \) and \( y = g(t) \). Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that \( x \) and \( y \) are both given as functions of a third variable (called a parameter) by the equations

\[
x = f(t) \quad \text{and} \quad y = g(t)
\]

(called parametric equations). Each value of \( t \) determines a point \((x, y)\), which we can plot in a coordinate plane. As \( t \) varies, the point \((x, y) = (f(t), g(t))\) varies and traces out a curve \( C \), which we call a parametric curve. The parameter \( t \) does not necessarily represent time and, in fact, we could use a letter other than \( t \) for the parameter. But in many applications of parametric curves, \( t \) does denote time and therefore we can interpret \((x, y) = (f(t), g(t))\) as the position of a particle at time \( t \).

**Example 1** Sketch and identify the curve defined by the parametric equations

\[
x = t^2 - 2t \quad \text{and} \quad y = t + 1
\]

**Solution** Each value of \( t \) gives a point on the curve, as shown in the table. For instance, if \( t = 0 \), then \( x = 0 \), \( y = 1 \) and so the corresponding point is \((0, 1)\). In Figure 2 we plot the points \((x, y)\) determined by several values of the parameter and we join them to produce a curve.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>8</td>
<td>-1</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>5</td>
</tr>
</tbody>
</table>

A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as \( t \) increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as \( t \) increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. This can be confirmed by eliminating the parameter \( t \) as follows. We obtain \( t = y - 1 \) from the second equation and substitute into the first equation. This gives

\[
x = t^2 - 2t = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3
\]

and so the curve represented by the given parametric equations is the parabola \( x = y^2 - 4y + 3 \).
No restriction was placed on the parameter \( t \) in Example 1, so we assumed that \( t \) could be any real number. But sometimes we restrict \( t \) to lie in a finite interval. For instance, the parametric curve

\[
x = t^2 - 2t \quad y = t + 1 \quad 0 \leq t \leq 4
\]

shown in Figure 3 is the part of the parabola in Example 1 that starts at the point \((0, 1)\) and ends at the point \((8, 5)\). The arrowhead indicates the direction in which the curve is traced as \( t \) increases from 0 to 4.

In general, the curve with parametric equations

\[
x = f(t) \quad y = g(t) \quad a \leq t \leq b
\]

has **initial point** \(( f(a), g(a))\) and **terminal point** \(( f(b), g(b))\).

**EXAMPLE 2** What curve is represented by the following parametric equations?

\[
x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi
\]

**SOLUTION** If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating \( t \). Observe that

\[
x^2 + y^2 = \cos^2 t + \sin^2 t = 1
\]

Thus the point \((x, y)\) moves on the unit circle \(x^2 + y^2 = 1\). Notice that in this example the parameter \( t \) can be interpreted as the angle (in radians) shown in Figure 4. As \( t \) increases from 0 to \( 2\pi \), the point \((x, y) = (\cos t, \sin t)\) moves once around the circle in the counterclockwise direction starting from the point \((1, 0)\).

**EXAMPLE 3** What curve is represented by the given parametric equations?

\[
x = \sin 2t \quad y = \cos 2t \quad 0 \leq t \leq 2\pi
\]

**SOLUTION** Again we have

\[
x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1
\]

so the parametric equations again represent the unit circle \(x^2 + y^2 = 1\). But as \( t \) increases from 0 to \( 2\pi \), the point \((x, y) = (\sin 2t, \cos 2t)\) starts at \((0, 1)\) and moves twice around the circle in the clockwise direction as indicated in Figure 5.

Examples 2 and 3 show that different sets of parametric equations can represent the same curve. Thus we distinguish between a curve, which is a set of points, and a parametric curve, in which the points are traced in a particular way.
EXAMPLE 4 Find parametric equations for the circle with center \((h, k)\) and radius \(r\).

SOLUTION If we take the equations of the unit circle in Example 2 and multiply the expressions for \(x\) and \(y\) by \(r\), we get \(x = r \cos t\), \(y = r \sin t\). You can verify that these equations represent a circle with radius \(r\) and center the origin traced counterclockwise. We now shift \(h\) units in the \(x\)-direction and \(k\) units in the \(y\)-direction and obtain parametric equations of the circle (Figure 6) with center \((h, k)\) and radius \(r\):

\[
x = h + r \cos t \quad y = k + r \sin t \quad 0 \leq t \leq 2\pi
\]

EXAMPLE 5 Sketch the curve with parametric equations \(x = \sin t\), \(y = \sin^2 t\).

SOLUTION Observe that \(y = (\sin t)^2 = x^2\) and so the point \((x, y)\) moves on the parabola \(y = x^2\). But note also that, since \(-1 \leq \sin t \leq 1\), we have \(-1 \leq x \leq 1\), so the parametric equations represent only the part of the parabola for which \(-1 \leq x \leq 1\). Since \(\sin t\) is periodic, the point \((x, y) = (\sin t, \sin^2 t)\) moves back and forth infinitely often along the parabola from \((-1, 1)\) to \((1, 1)\). (See Figure 7.)

\[\begin{array}{c}
\text{FIGURE 6} \\
x = h + r \cos t, y = k + r \sin t
\end{array}\]

\[\begin{array}{c}
\text{FIGURE 7}
\end{array}\]

**TEC** Module 10.1A gives an animation of the relationship between motion along a parametric curve \(x = f(t), y = g(t)\) and motion along the graphs of \(f\) and \(g\) as functions of \(t\). Clicking on TRIIG gives you the family of parametric curves

\[
x = a \cos bt \quad y = c \sin dt
\]

If you choose \(a = b = c = d = 1\) and click on animate, you will see how the graphs of \(x = \cos t\) and \(y = \sin t\) relate to the circle in Example 2. If you choose \(a = b = c = 1, d = 2\), you will see graphs as in Figure 8. By clicking on animate or moving the \(t\)-slider to the right, you can see from the color coding how motion along the graphs of \(x = \cos t\) and \(y = \sin 2t\) corresponds to motion along the parametric curve, which is called a Lissajous figure.

\[\begin{array}{c}
\text{FIGURE 8} \\
x = \cos t \quad y = \sin 2t \\
y = \sin 2t
\end{array}\]
GRAPHING DEVICES

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations. In fact, it’s instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

EXAMPLE 6 Use a graphing device to graph the curve \( x = y^4 - 3y^2 \).

SOLUTION If we let the parameter be \( t = y \), then we have the equations

\[
\begin{align*}
  x &= t^4 - 3t^2 \\
  y &= t
\end{align*}
\]

Using these parametric equations to graph the curve, we obtain Figure 9. It would be possible to solve the given equation \( (x = y^4 - 3y^2) \) for \( y \) as four functions of \( x \) and graph them individually, but the parametric equations provide a much easier method.

In general, if we need to graph an equation of the form \( x = g(y) \), we can use the parametric equations

\[
\begin{align*}
  x &= g(t) \\
  y &= t
\end{align*}
\]

Notice also that curves with equations \( y = f(x) \) (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

\[
\begin{align*}
  x &= t \\
  y &= f(t)
\end{align*}
\]

Graphing devices are particularly useful when sketching complicated curves. For instance, the curves shown in Figures 10, 11, and 12 would be virtually impossible to produce by hand.

One of the most important uses of parametric curves is in computer-aided design (CAD). In the Laboratory Project after Section 10.2 we will investigate special parametric curves, called Bézier curves, that are used extensively in manufacturing, especially in the automotive industry. These curves are also employed in specifying the shapes of letters and other symbols in laser printers.

THE CYCLOID

EXAMPLE 7 The curve traced out by a point \( P \) on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see Figure 13). If the circle has radius \( r \) and rolls along the \( x \)-axis and if one position of \( P \) is the origin, find parametric equations for the cycloid.
SOLUTION  We choose as parameter the angle of rotation $\theta$ of the circle ($\theta = 0$ when $P$ is at the origin). Suppose the circle has rotated through $\theta$ radians. Because the circle has been in contact with the line, we see from Figure 14 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of $P$ be $(x, y)$. Then from Figure 14 we see that

$$x = |OT| - |PQ| = r\theta - r \sin \theta = r(\theta - \sin \theta)$$

$$y = |TC| - |QC| = r - r \cos \theta = r(1 - \cos \theta)$$

Therefore parametric equations of the cycloid are

$$x = r(\theta - \sin \theta) \quad y = r(1 - \cos \theta) \quad \theta \in \mathbb{R}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \leq \theta \leq 2\pi$. Although Equations 1 were derived from Figure 14, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of $\theta$ (see Exercise 39).

Although it is possible to eliminate the parameter $\theta$ from Equations 1, the resulting Cartesian equation in $x$ and $y$ is very complicated and not as convenient to work with as the parametric equations.

One of the first people to study the cycloid was Galileo, who proposed that bridges be built in the shape of cycloids and who tried to find the area under one arch of a cycloid. Later this curve arose in connection with the brachistochrone problem: Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point $A$ to a lower point $B$ not directly beneath $A$. The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join $A$ to $B$, as in Figure 15, the particle will take the least time sliding from $A$ to $B$ if the curve is part of an inverted arch of a cycloid.

The Dutch physicist Huygens had already shown that the cycloid is also the solution to the tautochrone problem: that is, no matter where a particle $P$ is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 16). Huygens proposed that pendulum clocks (which he invented) swing in cycloidal arcs because then the pendulum takes the same time to make a complete oscillation whether it swings through a wide or a small arc.

FAMILIES OF PARAMETRIC CURVES

\[ \text{EXAMPLE 8} \]  Investigate the family of curves with parametric equations

$$x = a + \cos t \quad y = a \tan t + \sin t$$

What do these curves have in common? How does the shape change as $a$ increases?
CHAPTER 10 PARAMETRIC EQUATIONS AND POLAR COORDINATES

SOLUTION We use a graphing device to produce the graphs for the cases a = −2, −1, −0.5, −0.2, 0, 0.5, 1, and 2 shown in Figure 17. Notice that all of these curves (except the case a = 0) have two branches, and both branches approach the vertical asymptote x = a as x approaches a from the left or right.

![Graphs of curves with varying a values](image)

**FIGURE 17** Members of the family \( x = a + \cos t, \ y = a \tan t + \sin t, \) all graphed in the viewing rectangle \([-4, 4]\) by \([-4, 4]\)

When \( a < -1 \), both branches are smooth; but when \( a \) reaches \(-1\), the right branch acquires a sharp point, called a cusp. For a between \(-1\) and \(0\) the cusp turns into a loop, which becomes larger as \( a \) approaches \(0\). When \( a = 0\), both branches come together and form a circle (see Example 2). For a between \(0\) and \(1\), the left branch has a loop, which shrinks to become a cusp when \( a = 1\). For \( a > 1\), the branches become smooth again, and as \( a \) increases further, they become less curved. Notice that the curves with a positive tive are reflections about the -axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell.

### 10.1 EXERCISES

1–4 Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as \( t \) increases.

1. \( x = 1 + \sqrt{t}, \ y = t^2 - 4t, \ 0 \leq t \leq 5 \)
2. \( x = 2 \cos t, \ y = t - \cos t, \ 0 \leq t \leq 2\pi \)
3. \( x = 5 \sin t, \ y = t^2, \ -\pi \leq t \leq \pi \)
4. \( x = e^{-t} + t, \ y = e^t - t, \ -2 \leq t \leq 2 \)

5–10

(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as \( t \) increases.

(b) Eliminate the parameter to find a Cartesian equation of the curve.

5. \( x = 3t - 5, \ y = 2t + 1 \)
6. \( x = 1 + t, \ y = 5 - 2t, \ -2 \leq t \leq 3 \)
7. \( x = t^2 - 2, \ y = 5 - 2t, \ -3 \leq t \leq 4 \)

8. \( x = 1 + 3t, \ y = 2 - t^2 \)
9. \( x = \sqrt{t}, \ y = 1 - t \)
10. \( x = t^2, \ y = t^3 \)

11–18

(a) Eliminate the parameter to find a Cartesian equation of the curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

11. \( x = \sin \theta, \ y = \cos \theta, \ 0 \leq \theta \leq \pi \)
12. \( x = 4 \cos \theta, \ y = 5 \sin \theta, \ -\pi/2 \leq \theta \leq \pi/2 \)
13. \( x = \sin t, \ y = \csc t, \ 0 < t < \pi/2 \)
14. \( x = e^t - 1, \ y = e^t \)
15. \( x = e^t, \ y = t + 1 \)
16. \( x = \ln t, \ y = \sqrt{t}, \ t \geq 1 \)
17. \( x = \sinh t, \ y = \cosh t \)
18. \( x = 2 \cosh t, \quad y = 5 \sinh t \)

19–22 Describe the motion of a particle with position \((x, y)\) as \(t\) varies in the given interval.

19. \( x = 3 + 2 \cos t, \quad y = 1 + 2 \sin t, \quad \pi/2 \leq t \leq 3\pi/2 \)

20. \( x = 2 \sin t, \quad y = 4 + \cos t, \quad 0 \leq t \leq 3\pi/2 \)

21. \( x = 5 \sin t, \quad y = 2 \cos t, \quad -\pi \leq t \leq 5\pi \)

22. \( x = \sin t, \quad y = \cos^2 t, \quad -2\pi \leq t \leq 2\pi \)

23. Suppose a curve is given by the parametric equations \( x = f(t), \quad y = g(t) \), where the range of \( f \) is \([1, 4]\) and the range of \( g \) is \([2, 3]\). What can you say about the curve?

24. Match the graphs of the parametric equations \( x = f(t) \) and \( y = g(t) \) in (a)–(d) with the parametric curves labeled I–IV. Give reasons for your choices.

25–27 Use the graphs of \( x = f(t) \) and \( y = g(t) \) to sketch the parametric curve \( x = f(t), \quad y = g(t) \). Indicate with arrows the direction in which the curve is traced as \( t \) increases.

25.

26.

27.

28. Match the parametric equations with the graphs labeled I–VI. Give reasons for your choices. (Do not use a graphing device.)

(a) \( x = t^4 - t + 1, \quad y = t^2 \)

(b) \( x = t^2 - 2t, \quad y = \sqrt{t} \)

(c) \( x = \sin 2t, \quad y = \sin(t + \sin 2t) \)

(d) \( x = \cos 5t, \quad y = \sin 2t \)

(e) \( x = t + \sin 4t, \quad y = t^2 + \cos 3t \)

(f) \( x = \frac{\sin 2t}{4 + t^2}, \quad y = \frac{\cos 2t}{4 + t^2} \)

29. Graph the curve \( x = y - 3y^3 + y^5 \).

30. Graph the curves \( y = x^2 \) and \( x = y(y - 1)^2 \) and find their points of intersection correct to one decimal place.
31. (a) Show that the parametric equations
\[ x = x_1 + (x_2 - x_1)t \quad y = y_1 + (y_2 - y_1)t \]
where \( 0 \leq t \leq 1 \), describe the line segment that joins the points \( P_1(x_1, y_1) \) and \( P_2(x_2, y_2) \).
(b) Find parametric equations to represent the line segment from \((-2, 7)\) to \((3, -1)\).

32. Use a graphing device and the result of Exercise 31(a) to draw the triangle with vertices \((1, 1)\), \((4, 2)\), and \((1, 5)\).

33. Find parametric equations for the path of a particle that moves along the circle \( x^2 + (y - 1)^2 = 4 \) in the manner described.
(a) Once around clockwise, starting at \((2, 1)\)
(b) Three times around counterclockwise, starting at \((2, 1)\)
(c) Halfway around counterclockwise, starting at \((0, 3)\)

34. (a) Find parametric equations for the ellipse \( x^2/a^2 + y^2/b^2 = 1 \). [Hint: Modify the equations of the circle in Example 2.]
(b) Use these parametric equations to graph the ellipse when \( a = 3 \) and \( b = 1, 2, 4, \) and \( 8 \).
(c) How does the shape of the ellipse change as \( b \) varies?

35–36 Use a graphing calculator or computer to reproduce the picture.

37–38 Compare the curves represented by the parametric equations. How do they differ?
37. (a) \( x = t^3, \quad y = t^2 \)
(c) \( x = e^{-3t}, \quad y = e^{-2t} \)
38. (a) \( x = t, \quad y = t^{-2} \)
(c) \( x = e^t, \quad y = e^{-2t} \)

39. Derive Equations 1 for the case \( \pi/2 < \theta < \pi \).

40. Let \( P \) be a point at a distance \( d \) from the center of a circle of radius \( r \). The curve traced out by \( P \) as the circle rolls along a straight line is called a **trochoid**. (Think of the motion of a point on a spoke of a bicycle wheel.) The cycloid is the special case of a trochoid with \( d = r \). Using the same parameter \( \theta \) as for the cycloid and, assuming the line is the \( x \)-axis and \( \theta = 0 \) when \( P \) is at one of its lowest points, show that parametric equations of the trochoid are
\[ x = r \theta - d \sin \theta \quad y = r - d \cos \theta \]
Sketch the trochoid for the cases \( d < r \) and \( d > r \).

41. If \( a \) and \( b \) are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point \( P \) in the figure, using the angle \( \theta \) as the parameter. Then eliminate the parameter and identify the curve.

42. If \( a \) and \( b \) are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point \( P \) in the figure, using the angle \( \theta \) as the parameter. The line segment \( AB \) is tangent to the larger circle.

43. A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point \( P \) in the figure. Show that parametric equations for this curve can be written as
\[ x = 2a \cot \theta \quad y = 2a \sin^2 \theta \]
Sketch the curve.

44. (a) Find parametric equations for the set of all points \( P \) as shown in the figure such that \( |OP| = |AB| \). (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical method for constructing the edge of a cube whose volume is twice that of a given cube.)
(b) Use the geometric description of the curve to draw a rough sketch of the curve by hand. Check your work by using the parametric equations to graph the curve.

45. Suppose that the position of one particle at time \( t \) is given by
\[
x_1 = 3 \sin t \quad y_1 = 2 \cos t \quad 0 \leq t \leq 2\pi
\]
and the position of a second particle is given by
\[
x_2 = -3 + \cos t \quad y_2 = 1 + \sin t \quad 0 \leq t \leq 2\pi
\]
(a) Graph the paths of both particles. How many points of intersection are there?
(b) Are any of these points of intersection collision points?
(c) Describe what happens if the path of the second particle is given by
\[
x_2 = 3 + \cos t \quad y_2 = 1 + \sin t \quad 0 \leq t \leq 2\pi
\]

46. If a projectile is fired with an initial velocity of \( v_0 \) meters per second at an angle \( \alpha \) above the horizontal and air resistance is assumed to be negligible, then its position after \( t \) seconds is given by the parametric equations
\[
x = (v_0 \cos \alpha) t - \frac{1}{2}gt^2 \quad y = (v_0 \sin \alpha) t - \frac{1}{2}gt^2
\]
where \( g \) is the acceleration due to gravity (9.8 \( \text{m/s}^2 \)).
(a) If a gun is fired with \( \alpha = 30^\circ \) and \( v_0 = 500 \text{ m/s} \), when will the bullet hit the ground? How far from the ground will the bullet hit the ground? What is the maximum height reached by the bullet?
(b) Use a graphing device to check your answers to part (a). Then graph the path of the projectile for several other values of the angle \( \alpha \) to see where it hits the ground. Summarize your findings.
(c) Show that the path is parabolic by eliminating the parameter.

47. Investigate the family of curves defined by the parametric equations \( x = t^3, y = t^3 - ct \). How does the shape change as \( c \) increases? Illustrate by graphing several members of the family.

48. The swallowtail catastrophe curves are defined by the parametric equations \( x = 2ct - 4t^3, y = -ct^3 + 3t^4 \). Graph several of these curves. What features do the curves have in common? How do they change when \( c \) increases?

49. The curves with equations \( x = a \sin nt, y = b \cos t \) are called Lissajous figures. Investigate how these curves vary when \( a, b, \) and \( n \) vary. (Take \( n \) to be a positive integer.)

50. Investigate the family of curves defined by the parametric equations \( x = \cos t, y = \sin t - \sin ct, \) where \( c > 0 \). Start by letting \( c \) be a positive integer and see what happens to the shape as \( c \) increases. Then explore some of the possibilities that occur when \( c \) is a fraction.

Running Circles Around Circles

In this project we investigate families of curves, called hypocycloids and epicycloids, that are generated by the motion of a point on a circle that rolls inside or outside another circle.

1. A **hypocycloid** is a curve traced out by a fixed point \( P \) on a circle \( C \) of radius \( b \) as \( C \) rolls on the inside of a circle with center \( O \) and radius \( a \). Show that if the initial position of \( P \) is \( (a, 0) \) and the parameter \( \theta \) is chosen as in the figure, then parametric equations of the hypocycloid are
\[
x = (a - b) \cos \theta + b \cos \left( \frac{a - b}{b} \theta \right) \quad y = (a - b) \sin \theta - b \sin \left( \frac{a - b}{b} \theta \right)
\]

2. Use a graphing device (or the interactive graphic in TEC Module 10.1B) to draw the graphs of hypocycloids with \( a \) a positive integer and \( b = 1 \). How does the value of \( a \) affect the graph? Show that if we take \( a = 4 \), then the parametric equations of the hypocycloid reduce to
\[
x = 4 \cos^3 \theta \quad y = 4 \sin^3 \theta
\]
This curve is called a **hypocycloid of four cusps**, or an **astroid**.

[TEC] Look at Module 10.1B to see how hypocycloids and epicycloids are formed by the motion of rolling circles.
3. Now try \( b = 1 \) and \( a = n/d \), a fraction where \( n \) and \( d \) have no common factor. First let \( n = 1 \) and try to determine graphically the effect of the denominator \( d \) on the shape of the graph. Then let \( n \) vary while keeping \( d \) constant. What happens when \( n = d + 1 \)?

4. What happens if \( b = 1 \) and \( a \) is irrational? Experiment with an irrational number like \( \sqrt{2} \) or \( e - 2 \). Take larger and larger values for \( \theta \) and speculate on what would happen if we were to graph the hypocycloid for all real values of \( \theta \).

5. If the circle C rolls on the outside of the fixed circle, the curve traced out by \( P \) is called an epicycloid. Find parametric equations for the epicycloid.

6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2–4.

---

10.2

CALCULUS WITH PARAMETRIC CURVES

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, area, arc length, and surface area.

TANGENTS

In the preceding section we saw that some curves defined by parametric equations \( x = f(t) \) and \( y = g(t) \) can also be expressed, by eliminating the parameter, in the form \( y = F(x) \). (See Exercise 67 for general conditions under which this is possible.) If we substitute \( x = f(t) \) and \( y = g(t) \) in the equation \( y = F(x) \), we get

\[
g(t) = F(f(t))
\]

and so, if \( g, F, \) and \( f \) are differentiable, the Chain Rule gives

\[
g'(t) = F'(f(t))f'(t) = F'(x)f'(t)
\]

If \( f'(t) \neq 0 \), we can solve for \( F'(x) \):

\[
F'(x) = \frac{g'(t)}{f'(t)}
\]

Since the slope of the tangent to the curve \( y = F(x) \) at \( (x, F(x)) \) is \( F'(x) \), Equation 1 enables us to find tangents to parametric curves without having to eliminate the parameter. Using Leibniz notation, we can rewrite Equation 1 in an easily remembered form:

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} \quad \text{if} \quad \frac{dx}{dt} \neq 0
\]

It can be seen from Equation 2 that the curve has a horizontal tangent when \( \frac{dy}{dt} = 0 \) (provided that \( \frac{dx}{dt} \neq 0 \)) and it has a vertical tangent when \( \frac{dx}{dt} = 0 \) (provided that \( \frac{dy}{dt} \neq 0 \)). This information is useful for sketching parametric curves.
As we know from Chapter 4, it is also useful to consider \( \frac{d^2y}{dx^2} \). This can be found by replacing \( y \) by \( \frac{dy}{dx} \) in Equation 2:

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right)
\]

**EXAMPLE 1** A curve \( C \) is defined by the parametric equations \( x = t^2, y = t^3 - 3t \).
(a) Show that \( C \) has two tangents at the point \((3, 0)\) and find their equations.
(b) Find the points on \( C \) where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

**SOLUTION**

(a) Notice that \( y = t^3 - 3t = t(t^2 - 3) = 0 \) when \( t = 0 \) or \( t = \pm \sqrt{3} \). Therefore the point \((3, 0)\) on \( C \) arises from two values of the parameter, \( t = \sqrt{3} \) and \( t = -\sqrt{3} \). This indicates that \( C \) crosses itself at \((3, 0)\). Since

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2} \left( t - \frac{1}{t} \right)
\]

the slope of the tangent when \( t = \pm \sqrt{3} \) is \( \frac{dy}{dx} = \pm 6/(2 \sqrt{3}) = \pm \sqrt{3} \), so the equations of the tangents at \((3, 0)\) are

\[
y = \sqrt{3} (x - 3) \quad \text{and} \quad y = -\sqrt{3} (x - 3)
\]

(b) \( C \) has a horizontal tangent when \( \frac{dy}{dx} = 0 \), that is, when \( \frac{dy/dt}{dx/dt} = 0 \) and \( dx/dt \neq 0 \). Since \( \frac{dy/dt}{dx/dt} = 3t^2 - 3 \), this happens when \( t^2 = 1 \), that is, \( t = \pm 1 \). The corresponding points on \( C \) are \((1, -2)\) and \((1, 2)\). \( C \) has a vertical tangent when \( \frac{dx/dt}{dt} = 2t = 0 \), that is, \( t = 0 \). (Note that \( \frac{dy}{dt} \neq 0 \) there.) The corresponding point on \( C \) is \((0, 0)\).

(c) To determine concavity we calculate the second derivative:

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{3}{2} \left( \frac{1 + \frac{1}{t^2}}{2t} \right) = \frac{3(t^2 + 1)}{4t^3}
\]

Thus the curve is concave upward when \( t > 0 \) and concave downward when \( t < 0 \).

(d) Using the information from parts (b) and (c), we sketch \( C \) in Figure 1.

**EXAMPLE 2**

(a) Find the tangent to the cycloid \( x = r(\theta - \sin \theta), y = r(1 - \cos \theta) \) at the point where \( \theta = \pi/3 \). (See Example 7 in Section 10.1.)

(b) At what points is the tangent horizontal? When is it vertical?

**SOLUTION**

(a) The slope of the tangent line is

\[
\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r \sin \theta}{r(1 - \cos \theta)} = \frac{\sin \theta}{1 - \cos \theta}
\]
When \( \theta = \pi/3 \), we have

\[
x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \quad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}
\]

and

\[
\frac{dy}{dx} = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - \frac{1}{2}} = \sqrt{3}
\]

Therefore the slope of the tangent is \( \sqrt{3} \) and its equation is

\[
y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \quad \text{or} \quad \sqrt{3}x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)
\]

The tangent is sketched in Figure 2.

(b) The tangent is horizontal when \( \frac{dy}{dx} = 0 \), which occurs when \( \sin \theta = 0 \) and \( 1 - \cos \theta \neq 0 \), that is, \( \theta = (2n - 1)\pi \), \( n \) an integer. The corresponding point on the cycloid is \((2n - 1)\pi r, 2r\).

When \( \theta = 2n\pi \), both \( dx/d\theta \) and \( dy/d\theta \) are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l’Hospital’s Rule as follows:

\[
\lim_{\theta \to 2n_\pi^-} \frac{dy}{dx} = \lim_{\theta \to 2n_\pi^-} \frac{\sin \theta}{1 - \cos \theta} = \lim_{\theta \to 2n_\pi^-} \frac{\cos \theta}{\sin \theta} = \infty
\]

A similar computation shows that \( \frac{dy}{dx} \to -\infty \) as \( \theta \to 2n\pi^- \), so indeed there are vertical tangents when \( \theta = 2n\pi \), that is, when \( x = 2n\pi r \).

**AREAS**

We know that the area under a curve \( y = F(x) \) from \( a \) to \( b \) is \( A = \int_a^b F(x) \, dx \), where \( F(x) \geq 0 \). If the curve is traced out once by the parametric equations \( x = f(t) \) and \( y = g(t) \), \( \alpha \leq t \leq \beta \), then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

\[
A = \int_{\alpha}^{\beta} y \, dx = \int_{\alpha}^{\beta} g(t) \, f'(t) \, dt \quad \text{or} \quad \int_{\alpha}^{\beta} g(t) \, f'(t) \, dt
\]

**EXAMPLE 3** Find the area under one arch of the cycloid

\[
x = r(\theta - \sin \theta) \quad y = r(1 - \cos \theta)
\]

(See Figure 3.)
The result of Example 3 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (see Example 7 in Section 10.1). Galileo guessed this result but it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.

**FIGURE 3**

We already know how to find the length \( L \) of a curve \( C \) given in the form \( y = F(x) \), \( a \leq x \leq b \). Formula 8.1.3 says that if \( F' \) is continuous, then

\[
L = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

Suppose that \( C \) can also be described by the parametric equations \( x = f(t) \) and \( y = g(t) \), \( \alpha \leq t \leq \beta \), where \( \frac{dx}{dt} = f'(t) > 0 \). This means that \( C \) is traversed once, from left to right, as \( t \) increases from \( \alpha \) to \( \beta \) and \( f(\alpha) = a \), \( f(\beta) = b \). Putting Formula 2 into Formula 3 and using the Substitution Rule, we obtain

\[
L = \int_\alpha^\beta \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_\alpha^\beta \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \frac{dx}{dt} \, dt
\]

Since \( \frac{dx}{dt} > 0 \), we have

\[
L = \int_\alpha^\beta \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]

Even if \( C \) can’t be expressed in the form \( y = F(x) \), Formula 4 is still valid but we obtain it by polygonal approximations. We divide the parameter interval \([\alpha, \beta]\) into \( n \) subintervals of equal width \( \Delta t \). If \( t_0, t_1, t_2, \ldots, t_n \) are the endpoints of these subintervals, then \( x_i = f(t_i) \) and \( y_i = g(t_i) \) are the coordinates of points \( P_i(x_i, y_i) \) that lie on \( C \) and the polygon with vertices \( P_0, P_1, \ldots, P_n \) approximates \( C \). (See Figure 4.)

As in Section 8.1, we define the length \( L \) of \( C \) to be the limit of the lengths of these approximating polygons as \( n \to \infty \):

\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_{i-1}P_i|
\]

The Mean Value Theorem, when applied to \( f \) on the interval \([t_{i-1}, t_i]\), gives a number \( t^*_i \) in \((t_{i-1}, t_i)\) such that

\[
f(t_i) - f(t_{i-1}) = f'(t^*_i)(t_i - t_{i-1})
\]

If we let \( \Delta x_i = x_i - x_{i-1} \) and \( \Delta y_i = y_i - y_{i-1} \), this equation becomes

\[
\Delta x_i = f'(t^*_i) \Delta t
\]
Similarly, when applied to \( g \), the Mean Value Theorem gives a number \( t^* \) in \((t_{-1}, t_{})\) such that
\[
\Delta y_l = g'(t^*) \Delta t
\]
Therefore
\[
|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{[f'(t^*)]^2 + [g'(t^*)]^2} \Delta t
\]
and so
\[
L = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{[f'(t^*)]^2 + [g'(t^*)]^2} \Delta t
\]
The sum in (5) resembles a Riemann sum for the function \( \sqrt{[f'(t)]^2 + [g'(t)]^2} \) but it is not exactly a Riemann sum because \( t^* \neq t^{**} \) in general. Nevertheless, if \( f' \) and \( g' \) are continuous, it can be shown that the limit in (5) is the same as if \( t^* \) and \( t^{**} \) were equal, namely,
\[
L = \int_{\alpha}^{\beta} \sqrt{[f'(t)]^2 + [g'(t)]^2} \, dt
\]
Thus, using Leibniz notation, we have the following result, which has the same form as Formula (4).

**THEOREM** If a curve \( C \) is described by the parametric equations \( x = f(t) \), \( y = g(t) \), \( \alpha \leq t \leq \beta \), where \( f' \) and \( g' \) are continuous on \([\alpha, \beta]\) and \( C \) is traversed exactly once as \( t \) increases from \( \alpha \) to \( \beta \), then the length of \( C \) is
\[
L = \int_{\alpha}^{\beta} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt
\]
Notice that the formula in Theorem 6 is consistent with the general formulas \( L = \int ds \) and \( (ds)^2 = (dx)^2 + (dy)^2 \) of Section 8.1.

**EXAMPLE 4** If we use the representation of the unit circle given in Example 2 in Section 10.1,
\[
x = \cos t \quad y = \sin t \quad 0 \leq t \leq 2\pi
\]
then \( \frac{dx}{dt} = -\sin t \) and \( \frac{dy}{dt} = \cos t \), so Theorem 6 gives
\[
L = \int_{0}^{2\pi} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0}^{2\pi} \sqrt{\sin^2 t + \cos^2 t} \, dt = \int_{0}^{2\pi} \, dt = 2\pi
\]
as expected. If, on the other hand, we use the representation given in Example 3 in Section 10.1,
\[
x = \sin 2t \quad y = \cos 2t \quad 0 \leq t \leq 2\pi
\]
then \( \frac{dx}{dt} = 2 \cos 2t \), \( \frac{dy}{dt} = -2 \sin 2t \), and the integral in Theorem 6 gives
\[
\int_{0}^{2\pi} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_{0}^{2\pi} \sqrt{4 \cos^2 2t + 4 \sin^2 2t} \, dt = \int_{0}^{2\pi} 2 \, dt = 4\pi
\]
The result of Example 5 says that the length of one arch of the cycloid \( x = r(\theta - \sin \theta) \), \( y = r(1 - \cos \theta) \) proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul’s Cathedral in London.

### FIGURE 5

The integral gives twice the arc length of the circle because as \( t \) increases from 0 to \( 2\pi \), the point \((\sin 2t, \cos 2t)\) traverses the circle twice. In general, when finding the length of a curve \( C \) from a parametric representation, we have to be careful to ensure that \( C \) is traversed only once as \( t \) increases from \( \alpha \) to \( \beta \).

\[ L = \int_{\alpha}^{\beta} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

For parametric curves, we use\[ L = \int_{\alpha}^{\beta} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

### EXAMPLE 5

**SOLUTION** From Example 3 we see that one arch is described by the parameter interval \( 0 \leq \theta \leq 2\pi \). Since

\[ \frac{dx}{d\theta} = r(1 - \cos \theta) \quad \text{and} \quad \frac{dy}{d\theta} = r \sin \theta \]

we have

\[ L = \int_{0}^{2\pi} \sqrt{\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2} \, d\theta = \int_{0}^{2\pi} r \sqrt{1 - \cos^2 \theta + \sin^2 \theta} \, d\theta \]

\[ = \int_{0}^{2\pi} r \sqrt{2(1 - \cos \theta)} \, d\theta = 2r \int_{0}^{\pi} \sin(\theta/2) \, d\theta \]

and so

\[ L = 2r \left[ 2 \sin(\theta/2) \right]_{0}^{\pi} = 2r(2) = 4r \]

### SURFACE AREA

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. If the curve given by the parametric equations \( x = f(t), y = g(t), \alpha \leq t \leq \beta \), is rotated about the \( x \)-axis, where \( f', g' \) are continuous and \( g(t) \neq 0 \), then the area of the resulting surface is given by

\[ S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

The general symbolic formulas \( S = \int 2\pi y \, ds \) and \( S = \int 2\pi x \, ds \) (Formulas 8.2.7 and 8.2.8) are still valid, but for parametric curves we use

\[ ds = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt \]

### EXAMPLE 6

**SOLUTION** The sphere is obtained by rotating the semicircle

\[ x = r \cos t \quad y = r \sin t \quad 0 \leq t \leq \pi \]
about the x-axis. Therefore, from Formula 7, we get

\[ S = \int_0^\pi 2\pi r \sin t \sqrt{(-r \sin b)^2 + (r \cos b)^2} \, dt \]
\[ = 2\pi \int_0^\pi r \sin t \sqrt{r^2 \sin^2 t + \cos^2 t} \, dt = 2\pi \int_0^\pi r \sin t \cdot r \, dt \]
\[ = 2\pi r^2 \int_0^\pi \sin t \, dt = 2\pi r^2(-\cos t) \bigg|_0^\pi = 4\pi r^2 \]

### 10.2 Exercises

1–2 Find \(dy/dx\)

1. \(x = t \sin t, \quad y = t^2 + t\)
2. \(x = 1/t, \quad y = \sqrt{t}e^{-t}\)

3–6 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

3. \(x = t^4 + 1, \quad y = t^3 + t; \quad t = -1\)
4. \(x = t - t^{-1}, \quad y = 1 + t^2; \quad t = 1\)
5. \(x = e^t, \quad y = t - \ln t^2; \quad t = 1\)
6. \(x = \cos \theta + \sin 2\theta, \quad y = \sin \theta + \cos 2\theta; \quad \theta = 0\)

7–8 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

7. \(x = 1 + \ln t, \quad y = t^2 + 2; \quad (1, 3)\)
8. \(x = \tan \theta, \quad y = \sec \theta; \quad (1, \sqrt{2})\)

9–10 Find an equation of the tangent(s) to the curve at the given point. Then graph the curve and the tangent(s).

9. \(x = 6 \sin t, \quad y = t^2 + t; \quad (0, 0)\)
10. \(x = \cos t + \cos 2t, \quad y = \sin t + \sin 2t; \quad (-1, 1)\)

11–16 Find \(dy/dx\) and \(d^2y/dx^2\). For which values of \(t\) is the curve concave upward?

11. \(x = 4 + t^3, \quad y = t^2 + t^3\)
12. \(x = t^3 - 12t, \quad y = t^2 - 1\)
13. \(x = t - e^t, \quad y = t + e^{-t}\)
14. \(x = t + \ln t, \quad y = t - \ln t\)
15. \(x = 2 \sin t, \quad y = 3 \cos t, \quad 0 < t < 2\pi\)
16. \(x = \cos 2t, \quad y = \cos t, \quad 0 < t < \pi\)

17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

17. \(x = 10 - t^2, \quad y = t^3 - 12t\)
18. \(x = 2t^3 + 3t^2 - 12t, \quad y = 2t^3 + 3t^2 + 1\)

19. \(x = 2 \cos \theta, \quad y = \sin 2\theta\)
20. \(x = \cos 3\theta, \quad y = 2 \sin \theta\)

21. Use a graph to estimate the coordinates of the rightmost point on the curve \(x = t - t^6, \ y = e^t\). Then use calculus to find the exact coordinates.

22. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve \(x = t^4 - 2t, \ y = t + t^4\). Then find the exact coordinates.

23–24 Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

23. \(x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t\)
24. \(x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t\)

25. Show that the curve \(x = \cos t, \ y = \sin t \cos t\) has two tangents at \((0, 0)\) and find their equations. Sketch the curve.

26. Graph the curve \(x = \cos t + 2 \cos 2t, \ y = \sin t + 2 \sin 2t\) to discover where it crosses itself. Then find equations of both tangents at that point.

27. (a) Find the slope of the tangent line to the trochoid \(x = r(t) - d \sin \theta, \ y = r - d \cos \theta\) in terms of \(\theta\). (See Exercise 40 in Section 10.1.)
(b) Show that if \(d < r\), then the trochoid does not have a vertical tangent.

28. (a) Find the slope of the tangent to the astroid \(x = a \cos^3 \theta, \ y = a \sin^3 \theta\) in terms of \(\theta\). (Astroids are explored in the Laboratory Project on page 629.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?

29. At what points on the curve \(x = 2t^3, \ y = 1 + 4t - t^2\) does the tangent line have slope 1?

30. Find equations of the tangents to the curve \(x = 3t^2 + 1, \ y = 2t^3 + 1\) that pass through the point \((4, 3)\).

31. Use the parametric equations of an ellipse, \(x = a \cos \theta, \ y = b \sin \theta, \ 0 \leq \theta \leq 2\pi\), to find the area that it encloses.
32. Find the area enclosed by the curve \( x = t^2 - 2t, \ y = \sqrt{t} \) and the y-axis.
33. Find the area enclosed by the x-axis and the curve \( x = 1 + e^t, \ y = t - t^2 \).
34. Find the area of the region enclosed by the astroid \( x = a \cos^3 \theta, \ y = a \sin^3 \theta \). (Astroids are explored in the Laboratory Project on page 629.)
35. Find the area under one arch of the tropheid of Exercise 40 in Section 10.1 for the case \( d < r \).
36. Let \( R \) be the region enclosed by the loop of the curve in Example 1.
   (a) Find the area of \( R \).
   (b) If \( R \) is rotated about the x-axis, find the volume of the resulting solid.
   (c) Find the centroid of \( R \).

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.
37. \( x = t - t^2, \ y = \frac{1}{2}t^{3/2}, \ 1 \leq t \leq 2 \)
38. \( x = 1 + e^t, \ y = t^2, \ -3 \leq t \leq 3 \)
39. \( x = t + \cos t, \ y = t - \sin t, \ 0 \leq t \leq 2\pi \)
40. \( x = \ln t, \ y = \sqrt{t + 1}, \ 1 \leq t \leq 5 \)

41–44 Find the exact length of the curve.
41. \( x = 1 + 3t^2, \ y = 4 + 2t^3, \ 0 \leq t \leq 1 \)
42. \( x = e^t + e^{-t}, \ y = 5 - 2t, \ 0 \leq t \leq 3 \)
43. \( x = \frac{t}{1 + t}, \ y = \ln(1 + t), \ 0 \leq t \leq 2 \)
44. \( x = 3 \cos t - \cos 3t, \ y = 3 \sin t - \sin 3t, \ 0 \leq t \leq \pi \)

45–47 Graph the curve and find its length.
45. \( x = e^t \cos t, \ y = e^t \sin t, \ 0 \leq t \leq \pi \)
46. \( x = \cos t + \ln(\tan \frac{1}{2}t), \ y = \sin t, \ \pi/4 \leq t \leq 3\pi/4 \)
47. \( x = e^t - t, \ y = 4e^{t^2}, \ -8 \leq t \leq 3 \)
48. Find the length of the loop of the curve \( x = 3t - t^3, \ y = 3t^2 \).

49. Use Simpson’s Rule with \( n = 6 \) to estimate the length of the curve \( x = t - e^t, \ y = t + e^t, \ -6 \leq t \leq 6 \).
50. In Exercise 43 in Section 10.1 you were asked to derive the parametric equations \( x = 2a \cot \theta, \ y = 2a \sin^2 \theta \) for the curve called the witch of Maria Agnesi. Use Simpson’s Rule with \( n = 4 \) to estimate the length of the arc of this curve given by \( \pi/4 \leq \theta \leq \pi/2 \).

51–52 Find the distance traveled by a particle with position \((x, y)\) as \( t \) varies in the given time interval. Compare with the length of the curve.
51. \( x = \sin^2 t, \ y = \cos^3 t, \ 0 \leq t \leq 3\pi \)
52. \( x = \cos^2 t, \ y = \cos t, \ 0 \leq t \leq 4\pi \)

53. Show that the total length of the ellipse \( x = a \sin \theta, \ y = b \cos \theta, \ a > b > 0, \) is
   \[ L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta} \ d\theta \]
   where \( e \) is the eccentricity of the ellipse \((e = c/a, \) where \( c = \sqrt{a^2 - b^2} \)).
54. Find the total length of the astroid \( x = a \cos^3 \theta, \ y = a \sin^3 \theta, \) where \( a > 0 \).

55. (a) Graph the epitrochoid with equations
    \( x = 11 \cos t - 4 \cos(11t/2) \)
    \( y = 11 \sin t - 4 \sin(11t/2) \)
    What parameter interval gives the complete curve?
    (b) Use your CAS to find the approximate length of this curve.

56. A curve called Cornu’s spiral is defined by the parametric equations
    \( x = C(t) = \int_0^t \cos(\pi u^2/2) \ du \)
    \( y = S(t) = \int_0^t \sin(\pi u^2/2) \ du \)
    where \( C \) and \( S \) are the Fresnel functions that were introduced in Chapter 5.
    (a) Graph this curve. What happens as \( t \to \infty \) and as \( t \to -\infty \)?
    (b) Find the length of Cornu’s spiral from the origin to the point with parameter value \( t \).

57–58 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use your calculator to find the surface area correct to four decimal places.
57. \( x = 1 + te^t, \ y = (t^2 + 1)e^t, \ 0 \leq t \leq 1 \)
58. \( x = \sin^2 t, \ y = 3t^3, \ 0 \leq t \leq \pi/3 \)
59–61. Find the exact area of the surface obtained by rotating the given curve about the x-axis.

59. \( x = t^3, \quad y = t^4, \quad 0 \leq t \leq 1 \)

60. \( x = 3t - t^3, \quad y = 3t, \quad 0 \leq t \leq 1 \)

61. \( x = \cos^3 \theta, \quad y = \sin^3 \theta, \quad 0 \leq \theta \leq \pi/2 \)

62. Graph the curve
\[ x = 2 \cos \theta - \cos 2\theta, \quad y = 2 \sin \theta - \sin 2\theta \]
If this curve is rotated about the x-axis, find the area of the resulting surface. (Use your graph to help find the correct parameter interval.)

63. If the curve
\[ x = t + t^3, \quad y = t - \frac{1}{t^2}, \quad 1 \leq t \leq 2 \]
is rotated about the x-axis, use your calculator to estimate the area of the resulting surface to three decimal places.

64. If the arc of the curve in Exercise 50 is rotated about the x-axis, estimate the area of the resulting surface using Simpson’s Rule with \( n = 4 \).

65–66. Find the surface area generated by rotating the given curve about the y-axis.

65. \( x = 3t^2, \quad y = 2t^3, \quad 0 \leq t \leq 5 \)

66. \( x = e^t - t, \quad y = 4e^{-t^2}, \quad 0 \leq t \leq 1 \)

67. If \( f' \) is continuous and \( f'(t) \neq 0 \) for \( a \leq t \leq b \), show that the parametric curve \( x = f(t), \quad y = g(t), \quad a \leq t \leq b \), can be put in the form \( y = F(x) \). [Hint: Show that \( f^{-1} \) exists.]

68. Use Formula 2 to derive Formula 7 from Formula 8.2.5 for the case in which the curve can be represented in the form \( y = f(x) \), \( a \leq x \leq b \).

69. The curvature at a point \( P \) of a curve is defined as
\[ \kappa = \left| \frac{d\phi}{ds} \right| \]
where \( \phi \) is the angle of inclination of the tangent line at \( P \), as shown in the figure. Thus the curvature is the absolute value of the rate of change of \( \phi \) with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at \( P \) and will be studied in greater detail in Chapter 13.

(a) For a parametric curve \( x = x(t), \quad y = y(t) \), derive the formula
\[ \kappa = \left| \frac{\dot{x}y - \dot{y}x}{\dot{x}^2 + \dot{y}^2} \right|^{3/2} \]
where the dots indicate derivatives with respect to \( t \), so \( \dot{x} = dx/dt \). [Hint: Use \( \phi = \tan^{-1}(dy/dx) \) and Formula 2 to find \( d\phi/dt \). Then use the Chain Rule to find \( d\phi/ds \).]

(b) By regarding a curve \( y = f(x) \) as the parametric curve
\[ x = x, \quad y = f(x), \quad \text{with parameter} \quad x \]
show that the formula in part (a) becomes
\[ \kappa = \frac{|d^2y/dx^2|}{\left[ 1 + (dy/dx)^2 \right]^{3/2}} \]

70. (a) Use the formula in Exercise 69(b) to find the curvature of the parabola \( y = x^2 \) at the point \((1, 1)\).

(b) At what point does this parabola have maximum curvature?

71. Use the formula in Exercise 69(a) to find the curvature of the cycloid \( x = \theta - \sin \theta, \quad y = 1 - \cos \theta \) at the top of one of its arches.

72. (a) Show that the curvature at each point of a straight line is \( \kappa = 0 \).

(b) Show that the curvature at each point of a circle of radius \( r \) is \( \kappa = 1/r \).

73. A string is wound around a circle and then unwound while being held taut. The curve traced by the point \( P \) at the end of the string is called the involute of the circle. If the circle has radius \( r \) and center \( O \) and the initial position of \( P \) is \((r, 0)\), and if the parameter \( \theta \) is chosen as in the figure, show that parametric equations of the involute are
\[ x = r(\cos \theta + \theta \sin \theta), \quad y = r(\sin \theta - \theta \cos \theta) \]

74. A cow is tied to a silo with radius \( r \) by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.
A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the polar coordinate system, which is more convenient for many purposes.

We choose a point in the plane that is called the pole (or origin) and is labeled $O$. Then we draw a ray (half-line) starting at $O$ called the polar axis. This axis is usually drawn horizontally to the right and corresponds to the positive $x$-axis in Cartesian coordinates.

If $P$ is any other point in the plane, let $r$ be the distance from $O$ to $P$ and let $\theta$ be the angle (usually measured in radians) between the polar axis and the line $OP$ as in Figure 1. Then the point $P$ is represented by the ordered pair $(r, \theta)$ and $r, \theta$ are called polar coordinates of $P$. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of $\theta$.
We extend the meaning of polar coordinates \((r, \theta)\) to the case in which \(r\) is negative by agreeing that, as in Figure 2, the points \((-r, \theta)\) and \((r, \theta)\) lie on the same line through \(O\) and at the same distance \(|r|\) from \(O\), but on opposite sides of \(O\). If \(r > 0\), the point \((r, \theta)\) lies in the same quadrant as \(\theta\); if \(r < 0\), it lies in the quadrant on the opposite side of the pole. Notice that \((-r, \theta)\) represents the same point as \((r, \theta + \pi)\).

**EXAMPLE 1** Plot the points whose polar coordinates are given.

(a) \((1, 5\pi/4)\)  
(b) \((2, 3\pi)\)  
(c) \((2, -2\pi/3)\)  
(d) \((-3, 3\pi/4)\)

**SOLUTION** The points are plotted in Figure 3. In part (d) the point \((-3, 3\pi/4)\) is located three units from the pole in the fourth quadrant because the angle \(3\pi/4\) is in the second quadrant and \(r = -3\) is negative.

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point \((1, 5\pi/4)\) in Example 1(a) could be written as \((1, -3\pi/4)\) or \((1, 13\pi/4)\) or \((-1, \pi/4)\). (See Figure 4.)

In fact, since a complete counterclockwise rotation is given by an angle \(2\pi\), the point represented by polar coordinates \((r, \theta)\) is also represented by

\[(r, \theta + 2\pi n)\quad \text{and} \quad (-r, \theta + (2n + 1)\pi)\]

where \(n\) is any integer.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive \(x\)-axis. If the point \(P\) has Cartesian coordinates \((x, y)\) and polar coordinates \((r, \theta)\), then, from the figure, we have

\[\cos \theta = \frac{x}{r} \quad \text{and} \quad \sin \theta = \frac{y}{r}\]

and so

\[x = r \cos \theta \quad \text{and} \quad y = r \sin \theta\]

Although Equations 1 were deduced from Figure 5, which illustrates the case where \(r > 0\) and \(0 < \theta < \pi/2\), these equations are valid for all values of \(r\) and \(\theta\). (See the general definition of \(\sin \theta\) and \(\cos \theta\) in Appendix D.)
Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find \( r \) and \( \theta \) when \( x \) and \( y \) are known, we use the equations

\[
\begin{align*}
  r^2 &= x^2 + y^2 \\
  \tan \theta &= \frac{y}{x}
\end{align*}
\]

which can be deduced from Equations 1 or simply read from Figure 5.

**EXAMPLE 2** Convert the point \((2, \pi/3)\) from polar to Cartesian coordinates.

**SOLUTION** Since \( r = 2 \) and \( \theta = \pi/3 \), Equations 1 give

\[
\begin{align*}
  x &= r \cos \theta = 2 \cos \frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1 \\
  y &= r \sin \theta = 2 \sin \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}
\end{align*}
\]

Therefore the point is \((1, \sqrt{3})\) in Cartesian coordinates.

**EXAMPLE 3** Represent the point with Cartesian coordinates \((1, -1)\) in terms of polar coordinates.

**SOLUTION** If we choose \( r \) to be positive, then Equations 2 give

\[
\begin{align*}
  r &= \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2} \\
  \tan \theta &= \frac{y}{x} = -1
\end{align*}
\]

Since the point \((1, -1)\) lies in the fourth quadrant, we can choose \( \theta = -\pi/4 \) or \( \theta = 7\pi/4 \). Thus one possible answer is \((\sqrt{2}, -\pi/4)\); another is \((\sqrt{2}, 7\pi/4)\).

**NOTE** Equations 2 do not uniquely determine \( \theta \) when \( x \) and \( y \) are given because, as \( \theta \) increases through the interval \( 0 \leq \theta < 2\pi \), each value of \( \tan \theta \) occurs twice. Therefore, in converting from Cartesian to polar coordinates, it’s not good enough just to find \( r \) and \( \theta \) that satisfy Equations 2. As in Example 3, we must choose \( \theta \) so that the point \((r, \theta)\) lies in the correct quadrant.

**POLAR CURVES**

The **graph of a polar equation** \( r = f(\theta) \), or more generally \( F(r, \theta) = 0 \), consists of all points \( P \) that have at least one polar representation \((r, \theta)\) whose coordinates satisfy the equation.

**EXAMPLE 4** What curve is represented by the polar equation \( r = 2 \)?

**SOLUTION** The curve consists of all points \((r, \theta)\) with \( r = 2 \). Since \( r \) represents the distance from the point to the pole, the curve \( r = 2 \) represents the circle with center \( O \) and radius 2. In general, the equation \( r = a \) represents a circle with center \( O \) and radius \(|a|\). (See Figure 6.)
EXAMPLE 5 Sketch the polar curve $\theta = 1$.

SOLUTION This curve consists of all points $(r, \theta)$ such that the polar angle $\theta$ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis (see Figure 7). Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.

EXAMPLE 6
(a) Sketch the curve with polar equation $r = 2 \cos \theta$.
(b) Find a Cartesian equation for this curve.

SOLUTION
(a) In Figure 8 we find the values of $r$ for some convenient values of $\theta$ and plot the corresponding points $(r, \theta)$. Then we join these points to sketch the curve, which appears to be a circle. We have used only values of $\theta$ between 0 and $\pi$, since if we let $\theta$ increase beyond $\pi$, we obtain the same points again.

\[
\begin{array}{|c|c|}
\hline
\theta & r = 2 \cos \theta \\
\hline
0 & 2 \\
\pi/6 & \sqrt{3} \\
\pi/4 & \sqrt{2} \\
\pi/3 & 1 \\
\pi/2 & 0 \\
2\pi/3 & -1 \\
3\pi/4 & -\sqrt{2} \\
5\pi/6 & -\sqrt{3} \\
\pi & -2 \\
\hline
\end{array}
\]

(b) To convert the given equation to a Cartesian equation we use Equations 1 and 2. From $x = r \cos \theta$ we have $\cos \theta = x/r$, so the equation $r = 2 \cos \theta$ becomes $r = 2x/r$, which gives

\[2x = r^2 = x^2 + y^2 \quad \text{or} \quad x^2 + y^2 - 2x = 0\]

Completing the square, we obtain

\[(x - 1)^2 + y^2 = 1\]

which is an equation of a circle with center $(1, 0)$ and radius 1.
**EXAMPLE 7** Sketch the curve \( r = 1 + \sin \theta \).

**SOLUTION** Instead of plotting points as in Example 6, we first sketch the graph of \( r = 1 + \sin \theta \) in Cartesian coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of \( r \) that correspond to increasing values of \( \theta \). For instance, we see that as \( \theta \) increases from 0 to \( \pi/2 \), \( r \) (the distance from \( O \)) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a). As \( \theta \) increases from \( \pi/2 \) to \( \pi \), Figure 10 shows that \( r \) decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As \( \theta \) increases from \( \pi \) to \( 3\pi/2 \), \( r \) decreases from 1 to 0 as shown in part (c). Finally, as \( \theta \) increases from \( 3\pi/2 \) to \( 2\pi \), \( r \) increases from 0 to 1 as shown in part (d). If we let \( \theta \) increase beyond \( 2\pi \) or decrease beyond 0, we would simply retrace our path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid**, because it’s shaped like a heart.

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**EXAMPLE 8** Sketch the curve \( r = \cos 2\theta \).

**SOLUTION** As in Example 7, we first sketch \( r = \cos 2\theta \), \( 0 \leq \theta \leq 2\pi \), in Cartesian coordinates in Figure 12. As \( \theta \) increases from 0 to \( \pi/4 \), Figure 12 shows that \( r \) decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by (i)). As \( \theta \) increases from \( \pi/4 \) to \( \pi/2 \), \( r \) goes from 0 to \(-1\). This means that the distance from \( O \) increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by (ii)) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.