WRITING PROJECT

THE ORIGINS OF L'HOSPITAL'S RULE

www.stewartcalculus.com
The Internet is another source of information for this project. Click on *History of Mathematics* for a list of reliable websites.

L'Hospital's Rule was first published in 1696 in the Marquis de l'Hospital's calculus textbook *Analyse des Infiniment Petits,* but the rule was discovered in 1694 by the Swiss mathematician John (Johann) Bernoulli. The explanation is that these two mathematicians had entered into a curious business arrangement whereby the Marquis de l'Hospital bought the rights to Bernoulli's mathematical discoveries. The details, including a translation of l'Hospital's letter to Bernoulli proposing the arrangement, can be found in the book by Eves [1].

Write a report on the historical and mathematical origins of l'Hospital's Rule. Start by providing brief biographical details of both men (the dictionary edited by Gillispie [2] is a good source) and outline the business deal between them. Then give l'Hospital's statement of his rule, which is found in Struik's sourcebook [4] and more briefly in the book of Katz [3]. Notice that l'Hospital and Bernoulli formulated the rule geometrically and gave the answer in terms of differentials. Compare their statement with the version of l'Hospital's Rule given in Section 4.4 and show that the two statements are essentially the same.

1. Howard Eves, *In Mathematical Circles (Volume 2: Quadrants III and IV)* (Boston: Prindle, Weber and Schmidt, 1969), pp. 20–22.

2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Johann Bernoulli by E. A. Fellmann and J. O. Fleckenstein in Volume II and the article on the Marquis de l'Hospital by Abraham Robinson in Volume VIII.

3. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), p. 484.

4. D. J. Struik, ed., *A Sourcebook in Mathematics, 1200–1800* (Princeton, NJ: Princeton University Press, 1969), pp. 315–316.

4.5 SUMMARY OF CURVE SKETCHING

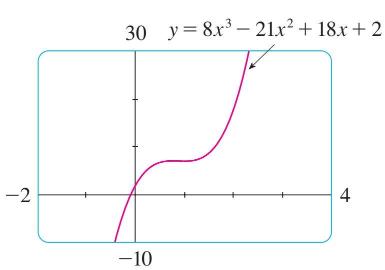


FIGURE 1

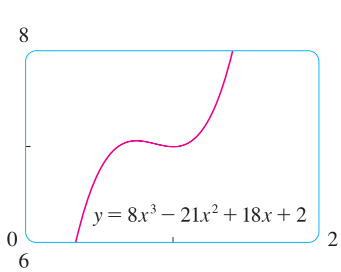


FIGURE 2

So far we have been concerned with some particular aspects of curve sketching: domain, range, and symmetry in Chapter 1; limits, continuity, and asymptotes in Chapter 2; derivatives and tangents in Chapters 2 and 3; and extreme values, intervals of increase and decrease, concavity, points of inflection, and l'Hospital's Rule in this chapter. It is now time to put all of this information together to sketch graphs that reveal the important features of functions.

You might ask: Why don't we just use a graphing calculator or computer to graph a curve? Why do we need to use calculus?

It's true that modern technology is capable of producing very accurate graphs. But even the best graphing devices have to be used intelligently. We saw in Section 1.4 that it is extremely important to choose an appropriate viewing rectangle to avoid getting a misleading graph. (See especially Examples 1, 3, 4, and 5 in that section.) The use of calculus enables us to discover the most interesting aspects of graphs and in many cases to calculate maximum and minimum points and inflection points *exactly* instead of approximately.

For instance, Figure 1 shows the graph of $f(x) = 8x^3 - 21x^2 + 18x + 2$. At first glance it seems reasonable: It has the same shape as cubic curves like $y = x^3$, and it appears to have no maximum or minimum point. But if you compute the derivative, you will see that there is a maximum when $x = 0.75$ and a minimum when $x = 1$. Indeed, if we zoom in to this portion of the graph, we see that behavior exhibited in Figure 2. Without calculus, we could easily have overlooked it.

In the next section we will graph functions by using the interaction between calculus and graphing devices. In this section we draw graphs by first considering the following information. We don't assume that you have a graphing device, but if you do have one you should use it as a check on your work.

GUIDELINES FOR SKETCHING A CURVE

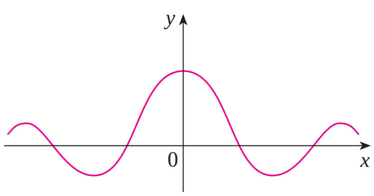
The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

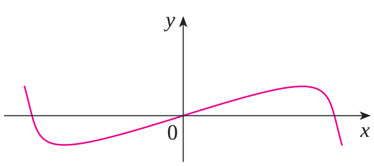
B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

FIGURE 3

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).

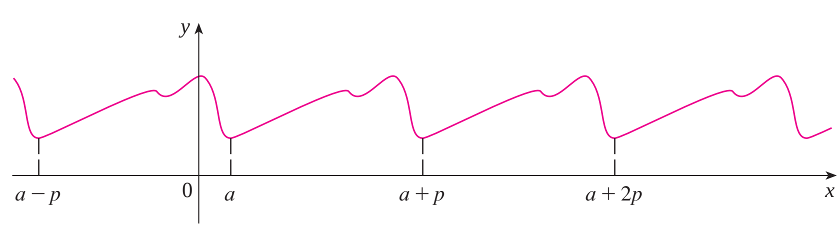


FIGURE 4
Periodic function: translational symmetry

D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 2.6 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

1

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.) Furthermore, in sketching the curve it is very useful to know exactly which of the statements in (1) is true. If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$, whether or not this limit is infinite.

(iii) *Slant Asymptotes.* These are discussed at the end of this section.

E. Intervals of Increase or Decrease Use the I/D Test. Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. Local Maximum and Minimum Values Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist]. Then use the First Derivative Test. If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum. If f' changes from negative to positive at c, then $f(c)$ is a local minimum. Although it is usually preferable to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$. Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection Compute $f''(x)$ and use the Concavity Test. The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$. Inflection points occur where the direction of concavity changes.

H. Sketch the Curve Using the information in items A–G, draw the graph. Sketch the asymptotes as dashed lines. Plot the intercepts, maximum and minimum points, and inflection points. Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes. If additional accuracy is desired near any point, you can compute the value of the derivative there. The tangent indicates the direction in which the curve proceeds.

V EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.

$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

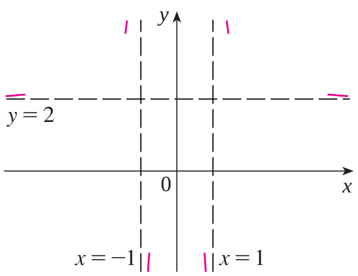


FIGURE 5
Preliminary sketch

■ We have shown the curve approaching its horizontal asymptote from above in Figure 5. This is confirmed by the intervals of increase and decrease.

E.
$$f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}$$

Since $f'(x) > 0$ when $x < 0$ ($x \neq -1$) and $f'(x) < 0$ when $x > 0$ ($x \neq 1$), f is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on $(0, 1)$ and $(1, \infty)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = 0$ is a local maximum by the First Derivative Test.

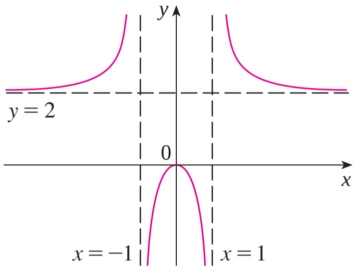


FIGURE 6
Finished sketch of $y = \frac{2x^2}{x^2 - 1}$

G.
$$f''(x) = \frac{-4(x^2 - 1)^2 + 4x \cdot 2(x^2 - 1)2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3}$$

Since $12x^2 + 4 > 0$ for all x, we have

$$f''(x) > 0 \iff x^2 - 1 > 0 \iff |x| > 1$$

and $f''(x) < 0 \iff |x| < 1$. Thus the curve is concave upward on the intervals $(-\infty, -1)$ and $(1, \infty)$ and concave downward on $(-1, 1)$. It has no point of inflection since 1 and -1 are not in the domain of f.

H. Using the information in E–G, we finish the sketch in Figure 6. ■

EXAMPLE 2 Sketch the graph of $f(x) = \dfrac{x^2}{\sqrt{x + 1}}$.

A. Domain $= \{x \mid x + 1 > 0\} = \{x \mid x > -1\} = (-1, \infty)$

B. The x- and y-intercepts are both 0.

C. Symmetry: None

D. Since

$$\lim_{x \to \infty} \frac{x^2}{\sqrt{x + 1}} = \infty$$

there is no horizontal asymptote. Since $\sqrt{x + 1} \to 0$ as $x \to -1^+$ and $f(x)$ is always positive, we have

$$\lim_{x \to -1^+} \frac{x^2}{\sqrt{x + 1}} = \infty$$

and so the line $x = -1$ is a vertical asymptote.

E.
$$f'(x) = \frac{2x\sqrt{x + 1} - x^2 \cdot 1/\left(2\sqrt{x + 1}\right)}{x + 1} = \frac{x(3x + 4)}{2(x + 1)^{3/2}}$$

We see that $f'(x) = 0$ when $x = 0$ (notice that $-\frac{4}{3}$ is not in the domain of f), so the only critical number is 0. Since $f'(x) < 0$ when $-1 < x < 0$ and $f'(x) > 0$ when $x > 0$, f is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$.

F. Since $f'(0) = 0$ and f' changes from negative to positive at 0, $f(0) = 0$ is a local (and absolute) minimum by the First Derivative Test.

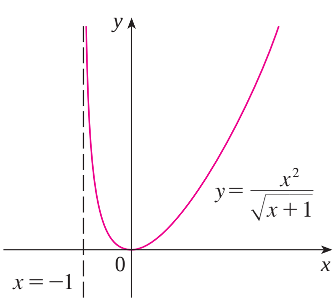


FIGURE 7

G.
$$f''(x) = \frac{2(x + 1)^{3/2}(6x + 4) - (3x^2 + 4x)3(x + 1)^{1/2}}{4(x + 1)^3} = \frac{3x^2 + 8x + 8}{4(x + 1)^{5/2}}$$

Note that the denominator is always positive. The numerator is the quadratic $3x^2 + 8x + 8$, which is always positive because its discriminant is $b^2 - 4ac = -32$, which is negative, and the coefficient of x^2 is positive. Thus $f''(x) > 0$ for all x in the domain of f, which means that f is concave upward on $(-1, \infty)$ and there is no point of inflection.

H. The curve is sketched in Figure 7. ■

V EXAMPLE 3 Sketch the graph of $f(x) = xe^x$.

A. The domain is $\mathbb{R}$.
B. The x- and y-intercepts are both 0.
C. Symmetry: None
D. Because both x and e^x become large as $x \to \infty$, we have $\lim_{x\to\infty} xe^x = \infty$. As $x \to -\infty$, however, $e^x \to 0$ and so we have an indeterminate product that requires the use of l'Hospital's Rule:

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} (-e^x) = 0$$

Thus the x-axis is a horizontal asymptote.

E. $$f'(x) = xe^x + e^x = (x+1)e^x$$

Since e^x is always positive, we see that $f'(x) > 0$ when $x + 1 > 0$, and $f'(x) < 0$ when $x + 1 < 0$. So f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.
F. Because $f'(-1) = 0$ and f' changes from negative to positive at $x = -1$, $f(-1) = -e^{-1}$ is a local (and absolute) minimum.
G. $$f''(x) = (x+1)e^x + e^x = (x+2)e^x$$

Since $f''(x) > 0$ if $x > -2$ and $f''(x) < 0$ if $x < -2$, f is concave upward on $(-2, \infty)$ and concave downward on $(-\infty, -2)$. The inflection point is $(-2, -2e^{-2})$.
H. We use this information to sketch the curve in Figure 8. ■

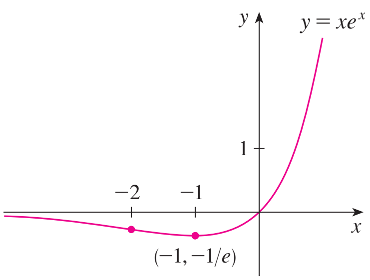


FIGURE 8

EXAMPLE 4 Sketch the graph of $f(x) = \dfrac{\cos x}{2 + \sin x}$.

A. The domain is $\mathbb{R}$.
B. The y-intercept is $f(0) = \frac{1}{2}$. The x-intercepts occur when $\cos x = 0$, that is, $x = (2n+1)\pi/2$, where n is an integer.
C. f is neither even nor odd, but $f(x + 2\pi) = f(x)$ for all x and so f is periodic and has period 2π. Thus, in what follows, we need to consider only $0 \leqslant x \leqslant 2\pi$ and then extend the curve by translation in part H.
D. Asymptotes: None
E. $$f'(x) = \frac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = -\frac{2\sin x + 1}{(2 + \sin x)^2}$$

Thus $f'(x) > 0$ when $2\sin x + 1 < 0 \iff \sin x < -\frac{1}{2} \iff 7\pi/6 < x < 11\pi/6$. So f is increasing on $(7\pi/6, 11\pi/6)$ and decreasing on $(0, 7\pi/6)$ and $(11\pi/6, 2\pi)$.
F. From part E and the First Derivative Test, we see that the local minimum value is $f(7\pi/6) = -1/\sqrt{3}$ and the local maximum value is $f(11\pi/6) = 1/\sqrt{3}$.
G. If we use the Quotient Rule again and simplify, we get

$$f''(x) = -\frac{2\cos x\,(1 - \sin x)}{(2 + \sin x)^3}$$

Because $(2 + \sin x)^3 > 0$ and $1 - \sin x \geqslant 0$ for all x, we know that $f''(x) > 0$ when $\cos x < 0$, that is, $\pi/2 < x < 3\pi/2$. So f is concave upward on $(\pi/2, 3\pi/2)$ and concave downward on $(0, \pi/2)$ and $(3\pi/2, 2\pi)$. The inflection points are $(\pi/2, 0)$ and $(3\pi/2, 0)$.

H. The graph of the function restricted to $0 \leqslant x \leqslant 2\pi$ is shown in Figure 9. Then we extend it, using periodicity, to the complete graph in Figure 10.

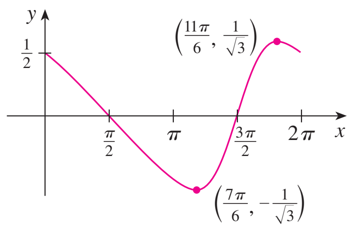


FIGURE 9

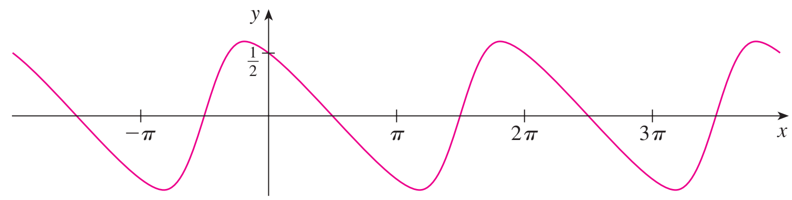


FIGURE 10

EXAMPLE 5 Sketch the graph of $y = \ln(4 - x^2)$.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x \to -2^+} \ln(4 - x^2) = -\infty$$

Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E.
$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

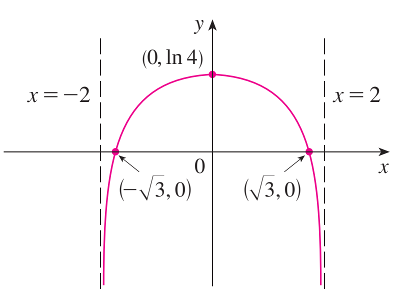


FIGURE 11
$y = \ln(4 - x^2)$

H. Using this information, we sketch the curve in Figure 11. ■

SLANT ASYMPTOTES

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x \to \infty} [f(x) - (mx + b)] = 0$$

then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance

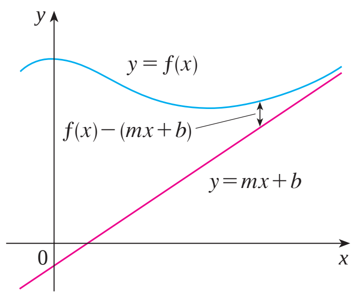


FIGURE 12

between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure 12. (A similar situation exists if we let $x \to -\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.

V EXAMPLE 6 Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

A. The domain is $\mathbb{R} = (-\infty, \infty)$.
B. The x- and y-intercepts are both 0.
C. Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.
D. Since $x^2 + 1$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

$$f(x) = \frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$$

$$f(x) - x = -\frac{x}{x^2 + 1} = -\frac{\frac{1}{x}}{1 + \frac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $y = x$ is a slant asymptote.

E. $$f'(x) = \frac{3x^2(x^2 + 1) - x^3 \cdot 2x}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}$$

Since $f'(x) > 0$ for all x (except 0), f is increasing on $(-\infty, \infty)$.
F. Although $f'(0) = 0$, f' does not change sign at 0, so there is no local maximum or minimum.

G. $$f''(x) = \frac{(4x^3 + 6x)(x^2 + 1)^2 - (x^4 + 3x^2) \cdot 2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{2x(3 - x^2)}{(x^2 + 1)^3}$$

Since $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{3}$, we set up the following chart:

Interval	x	$3 - x^2$	$(x^2 + 1)^3$	$f''(x)$	f
$x < -\sqrt{3}$	−	−	+	+	CU on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	−	+	+	−	CD on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	+	+	+	+	CU on $(0, \sqrt{3})$
$x > \sqrt{3}$	+	−	+	−	CD on $(\sqrt{3}, \infty)$

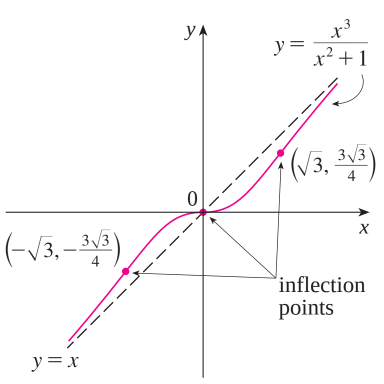


FIGURE 13

The points of inflection are $\left(-\sqrt{3}, -\frac{3}{4}\sqrt{3}\right)$, $(0, 0)$, and $\left(\sqrt{3}, \frac{3}{4}\sqrt{3}\right)$.
H. The graph of f is sketched in Figure 13. ■

4.5 EXERCISES

1–52 Use the guidelines of this section to sketch the curve.

1. $y = x^3 + x$

2. $y = x^3 + 6x^2 + 9x$

3. $y = 2 - 15x + 9x^2 - x^3$

4. $y = 8x^2 - x^4$

5. $y = x^4 + 4x^3$

6. $y = x(x + 2)^3$

7. $y = 2x^5 - 5x^2 + 1$

8. $y = (4 - x^2)^5$

9. $y = \dfrac{x}{x - 1}$

10. $y = \dfrac{x^2 - 4}{x^2 - 2x}$

11. $y = \dfrac{1}{x^2 - 9}$

12. $y = \dfrac{x}{x^2 - 9}$

13. $y = \dfrac{x}{x^2 + 9}$

14. $y = \dfrac{x^2}{x^2 + 9}$

15. $y = \dfrac{x - 1}{x^2}$

16. $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

17. $y = \dfrac{x^2}{x^2 + 3}$

18. $y = \dfrac{x}{x^3 - 1}$

19. $y = x\sqrt{5 - x}$

20. $y = 2\sqrt{x} - x$

21. $y = \sqrt{x^2 + x - 2}$

22. $y = \sqrt{x^2 + x} - x$

23. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

24. $y = x\sqrt{2 - x^2}$

25. $y = \dfrac{\sqrt{1 - x^2}}{x}$

26. $y = \dfrac{x}{\sqrt{x^2 - 1}}$

27. $y = x - 3x^{1/3}$

28. $y = x^{5/3} - 5x^{2/3}$

29. $y = \sqrt[3]{x^2 - 1}$

30. $y = \sqrt[3]{x^3 + 1}$

31. $y = 3\sin x - \sin^3 x$

32. $y = x + \cos x$

33. $y = x\tan x, \quad -\pi/2 < x < \pi/2$

34. $y = 2x - \tan x, \quad -\pi/2 < x < \pi/2$

35. $y = \frac{1}{2}x - \sin x, \quad 0 < x < 3\pi$

36. $y = \sec x + \tan x, \quad 0 < x < \pi/2$

37. $y = \dfrac{\sin x}{1 + \cos x}$

38. $y = \dfrac{\sin x}{2 + \cos x}$

39. $y = e^{\sin x}$

40. $y = e^{-x}\sin x, \quad 0 \leqslant x \leqslant 2\pi$

41. $y = 1/(1 + e^{-x})$

42. $y = e^{2x} - e^x$

43. $y = x - \ln x$

44. $y = e^x/x$

45. $y = (1 + e^x)^{-2}$

46. $y = \ln(x^2 - 3x + 2)$

47. $y = \ln(\sin x)$

48. $y = \dfrac{\ln x}{x^2}$

49. $y = xe^{-x^2}$

50. $y = (x^2 - 3)e^{-x}$

51. $y = e^{3x} + e^{-2x}$

52. $y = \tan^{-1}\left(\dfrac{x - 1}{x + 1}\right)$

53. In the theory of relativity, the mass of a particle is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v relative to the observer, and c is the speed of light. Sketch the graph of m as a function of v.

54. In the theory of relativity, the energy of a particle is

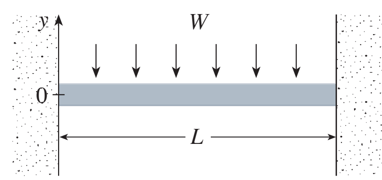
$$E = \sqrt{m_0^2c^4 + h^2c^2/\lambda^2}$$

where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What does the graph say about the energy?

55. The figure shows a beam of length L embedded in concrete walls. If a constant load W is distributed evenly along its length, the beam takes the shape of the deflection curve

$$y = -\frac{W}{24EI}x^4 + \frac{WL}{12EI}x^3 - \frac{WL^2}{24EI}x^2$$

where E and I are positive constants. (E is Young's modulus of elasticity and I is the moment of inertia of a cross-section of the beam.) Sketch the graph of the deflection curve.



56. Coulomb's Law states that the force of attraction between two charged particles is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The figure shows particles with charge 1 located at positions 0 and 2 on a coordinate line and a particle with charge -1 at a position x between them. It follows from Coulomb's Law that the net force acting on the middle particle is

$$F(x) = -\frac{k}{x^2} + \frac{k}{(x - 2)^2} \qquad 0 < x < 2$$

where k is a positive constant. Sketch the graph of the net force function. What does the graph say about the force?