**Discrete Time Markov Chain (DTMC)**

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**Fair play – tossing a coin**

Fair coin is tossed repeatedly.  
Gambler bets on result “head” one euro. (win with probability $p = 1/2$).  
Let us denote the state of this stochastic process by gamblers amount of money. Than state space $S$ is countable set. $S = \{\ldots, -1, -2, 0, 1, 2, \ldots\}$.  
Discretization of a time: 1 step = 1 game  
Simulation of 100 games:

```matlab
steps=100;  
S=zeros(steps,1)  
for i =1:steps-1  
    if rand>0.5  
        S(i+1)=S(i)+1  
    else  
        S(i+1)=S(i)-1  
    end  
end  
plot(1:steps,S,'*m--')
```
Simulation – repeated play

Repeat the experiment for 1000 gamblers, determine the distribution of final win.

```
function [win] = game(steps,p)
    % [S(i)]=game(steps,p)%No steps, probability of success
    S = zeros(steps,1);
    for i = 1:steps-1
        if rand < p
            S(i+1) = S(i) + 1;
        else
            S(i+1) = S(i) - 1;
        end
    end
    win = S(end);
end
```

```
[h,p] = chi2gof(P)
```

Tossing a coin - Finite Game

```
% S = [-2,1,0,1,2]
function [vyhra,S] = game2(steps,p)
    % [S(i)]=game2(steps,p)%No steps, prob. Of success
    S(1) = 0;
    for i = 1:steps-1
        if rand < p
            S(i+1) = S(i) + 1;
        else
            S(i+1) = S(i) - 1;
        end
        if S(i+1) == 2 || S(i+1) == -2, break
    end
end
win = S(length(S));
end
```
Game as 1D random walk

Tossing a coin is an example of random walk: If we picture all states arranged on line, we move from one state to one of its neighbours.

- How process evolves depends on the probability of moving from one state to another.
- The transition matrix $P$ is a matrix whose $(i,j)$th entry is transition probabilities $p_{ij}$ from $i$th state to $j$th state.

$$p_{i, i+1} = p$$
$$p_{i+1, i} = 1 - p$$

Finite game

Modification: The chain will terminate at states -2 or 2– first player or the second player wins.

$$P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 - p & 0 & p & 0 & 0 \\
0 & 1 - p & 0 & p & 0 \\
0 & 0 & 1 - p & 0 & p \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$
Tossing coin as a Markov chain

**Memoryless property**
The outcome of the $n$th toss is independent of the results of tosses 1,..., $n-1$.
If we know what happened at time $n-1$, then any other information about the past does not affect the probability distribution for time $n$.

In general, a Markov chain is given by
- A state space $S$ – a countable set of states
- Transition probabilities $p_{ij}$
- Initial distribution $\alpha(0)$

Finite game

$$\begin{pmatrix}
-2 & -1 & 0 & 1 & 2 \\
1 & 0 & 0 & 0 & 0 \\
1 - p & 0 & p & 0 & 0 \\
0 & 1 - p & 0 & p & 0 \\
0 & 0 & 1 - p & 0 & p
\end{pmatrix}$$

Initial distribution $\alpha(0) = (0,0,1,0,0)$
Prob. distribution after 1 game: $\alpha(1) = (0,1-p,0,p,0) = \alpha(0)P$
Prob. distribution after 2 games: $\alpha(2) = \alpha(1)P = \alpha(0)P^2$

... 
Prob. distribution after $n$ games: $\alpha(n) = \alpha(n - 1)P = \alpha(0)P^n$

Probability distribution vector $\alpha(n)$ denotes the probabilities that the system is in each state at time $n$. Probability distribution vector $\alpha(n)$ depends upon the initial state of the system $\alpha(0)$ and transition matrix $P$. 
Transition matrix

Square matrix $P$, $p_{ij}$ means probability of transition from place $P_i$ to place $P_j$.

$$P = \begin{bmatrix}
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 & 0 \\
0 & 0 & 0 & 0.5 & 0.5 \\
0.5 & 0 & 0 & 0 & 0.5 \\
0.5 & 0.5 & 0 & 0 & 0
\end{bmatrix}$$

$a(1) = a(0)P$

$a(2) = a(1)P$

$\vdots$

$a(n+1) = a(n)P$

$$\lim_{n \to \infty} a(n+1) = \lim_{n \to \infty} a(n) = a = aP$$

\[
a = aP
\]

\[
a(P - E) = 0
\]

\[
a \left\{ P - E \right\} = (0, \ldots, 0, 1)
\]

```
clear;
P=[0,0.5,0.5,0;0,0,0.5,0.5,0;0,0,0,0.5,...
       0.5;0.5,0,0,0.5;0.5,0.5,0,0,0]
P^20
A=([P-eye(5));[1;1;1;1;1]); %normalization
a=[0,0,0,0,1]/A % A.a=[0,0,0,1]
```
Example – Stochastic Petri Net

- **State space** = \{P_1, P_2, P_3\}
- **Transition matrix** \( P \)
  - square matrix \( P \), \( p_{ij} \) means probability of transition from place \( P_i \) to place \( P_j \).
  - Initial distribution \( a(0) \) and transition matrix \( P \) uniquely determine the distribution \( a(n) \) after \( n \) steps.

\[
P = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 2 \\
1 & 0 & 0
\end{pmatrix}
\]

\[
a(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}
\]

\[
a(1) \cdot P = a(0)
\]

Markov Chains

- A Markov Chain is a weighted digraph representing a discrete-time system that can be in any number of discrete states.
- The transition matrix \( P \) for a Markov chain is matrix of probabilities of moving from one state to another.
- \( p_{ij} \) = probability of moving from state \( i \) to \( j \) is **independent of what happened before** moving to state \( j \) and how one got to state \( i \) (Markov assumption).
- Sum of probabilities \( \alpha(n) \) for each time \( n \) must be one.

\[
\alpha(n) = \alpha(n - 1)P = \alpha(0)P^n
\]
Birth-death chain

Example 12.8. Birth-death chain. This is a general model in which a population may change by at most 1 at each time step. Assume the size of a population is \( x \). The birth probability \( p_x \) is the transition probability to \( x + 1 \), the death probability \( q_x \) is the transition to \( x - 1 \), and \( r_x = 1 - p_x - q_x \) is the transition to \( x \). Clearly, \( q_0 = 0 \). The transition matrix is now

\[
\begin{bmatrix}
  r_0 & p_0 & 0 & 0 & 0 & \cdots \\
  q_1 & r_1 & p_1 & 0 & 0 & \cdots \\
  0 & q_2 & r_2 & p_2 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

Stable (regular) process

- A common question arising in Markov-chain models is, what is the long-term probability that the system will be in each state?
- The vector containing these long-term probabilities is called the steady-state vector of the Markov chain

- Stable process – \( \alpha(n) \) tends to a limit \( \alpha \) as \( n \to \infty \), this steady state limit does not depend on the initial state.
- Key to the study of Markov chains is the study of powers of transition matrix \( P \).
- Any nonnegative vector which satisfies \( \alpha = \alpha P \) and whose components sum to one is called a stationary probability distribution of the Markov Chain.
11.4.2016

How the Game is Played

• Chutes and Ladders is a board game where players spin a pointer to determine how they will advance
• The board consists of 100 numbered squares
• The objective is to land on square 100
• However, the board is filled with chutes and ladders, which move a player backward or forward if landed on.

Simulation

• Objectives:
  – Find frequencies for being at each position
  – Find mean number of moves to win
  – Find standard deviation
  – Simulate a large number of games
Results of Transition Matrix

- After 1000 moves, the probability vector reached a limit of \( \{0,0,\ldots,1\} \).
- This means that after 1000 moves, the game is expected to be won!!!

Results of simulation

- We ran 250,000 games
- Mean is approximately 39.65 moves to reach square 100.
- The standard deviation is approximately 24.00

DTMC simulation

DTMC with 3 states is given by transition matrix \( P \) and initial state \( \alpha(1) = (1,0,0) \).
1. Simulate one run with 100 steps. (resp. 500, 1000 steps)
2. Estimate the steady-state distribution

\[
P = \begin{pmatrix}
0.2 & 0.3 & 0.5 \\
0.1 & 0.1 & 0.8 \\
0.4 & 0.3 & 0.3
\end{pmatrix}
\]

```matlab
P=[0.2,0.3,0.5;0.1,0.1,0.8;0.4,0.3,0.3];
steps=100;
states=zeros(steps,1)
states(1)=1% start,
for i=1:steps-1
    states(i+1)=find(rand<cumsum(P(states(i,:)),1),1)
end
%steady-state distribution
freq=hist(states,1:3);
a_est=freq/steps
```

DTMC.m
DTMC analytical solution

DTMC with 3 states is given by transition matrix $P$ and initial state $\alpha(1) = (1,0,0)$.

1. Determine $P^n$
2. Determine the steady-state distribution

```matlab
% Steady-state distribution 
lim a(n) = a(0) P^n ever since P^8 
% a = (0.2708, 0.2500, 0.4792) 
lim P = P^10 
A = [(P-eye(length(P)))'; [1,1,1]] 
a = A \{0;0;1\} % aa=[0,0,1]' together with normalize
```

Bernoulli trials as Markov chain

There are only two possible outcomes for each trial, often designated success or failure. The probability of success, $p$, is the same for every trial.

State space = $(S, F)$

$$P = \begin{pmatrix}
p & 1-p \\
p & 1-p 
\end{pmatrix}$$

Probability vector $a(n)$ - probability of states $S$ or $F$ after $n$ trials.

$$(a_S(1), a_F(1)) = (p a_S(0) + a_F(0), (1-p)(a_S(0) + a_F(0))) = (p, 1-p)$$

$$a = aP$$

$$(a_s, a_f) = (p(a_s + a_f), (1-p)(a_s + a_f))$$

$$(a_s, a_f) = (p, 1-p)$$

Bernoulli.m
Geometric distribution as absorbing Markov chain

The geometric distribution is the only discrete memoryless random distribution. It is a discrete analog of the exponential distribution.

The probability distribution of the number $X$ of Bernoulli trials needed to get one success

$$P = \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix}; \quad a(0) = (0.1)$$

An absorbing state is a state that, once entered, cannot be left.

$$a = aP$$

$$(a_s, a_f) = (a_s + pa_f, (1-p)a_f)$$

$$(a_s, a_f) = (1.0)$$

Geometric distribution (control and security measures)

Example: Functioning of some device is inspected once a day – at 6p.m. Probability $p$ of defect (success) is same for the whole of observed time: $p = 0.1$.

Estimate the probability that errorless period is longer then 5 days.

Estimate the average errorless period.

$$a(0) = (0.1)$$

$$a(n) = (1-(1-p)^n, (1-p)^n)$$

$$p = 0.1$$

$$a(6) = (1-0.9^6; 0.9^6)$$

$$1 - cdf(6) = (1-p)^6 = 0.53$$

$$E[X] = \frac{1-p}{p} = 9(days)$$