# Discrete Time Markov Chain (DTMC)



# Fair play – tossing a coin

Fair coin is tossed repeatedly.

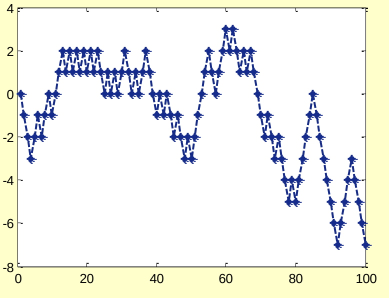
Gambler bets on result "head" one euro. (win with probability $p = 1/2$).

Let us denote the state of this stochastic process by gamblers amount of money. Than state space S is countable set. $S = \{\dots, -1, -2, 0, 1, 2, \dots\}$.

Discretization of a time: 1 step = 1 game

Simulation of 100 games:

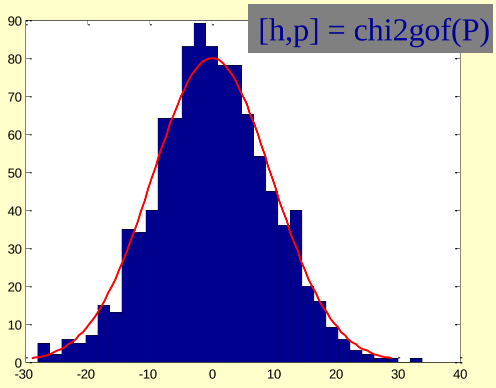
```
steps=100;
S=zeros(steps,1)
for i =1:steps-1
    if rand>0.5
    S(i+1)=S(i)+1
    else
        S(i+1)=S(i)-1
    end
end
plot(1:steps,S,'*m--')
```



# Simulation – repeated play

Repeat the experiment for 1000 gamblers, determine the distribution of final win.

```
function[win]=game(steps,p)
% [S(i)]=game(steps,p)%No steps, probability of success
S=zeros(steps,1);
for i =1:steps-1
    if rand<p
    S(i+1)=S(i)+1;
    else
       S(i+1)=S(i)-1;
    end
end
win=S(i);
end
```




[h,p] = chi2gof(P)

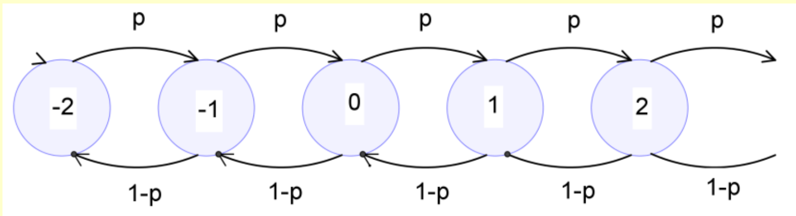
# Tossing a coin - Finite Game

% S = {-2,-1,0,1,2}

```
function[vyhra,S]=game2(steps,p)
% [S(i)]=game2(steps,p)%No steps, prob. Of success
S(1)=0;
for i =1:steps-1
    if rand<p
    S(i+1)=S(i)+1;
    else
       S(i+1)=S(i)-1;
    end
    if S(i+1)== 2 || S(i+1)== -2, break
    end
end
win=S(length(S));
end
```

# Game as 1D random walk

Tossing a coin is an example of random walk: If we picture all states arranged on line, we move from one state to one of its neighbours.

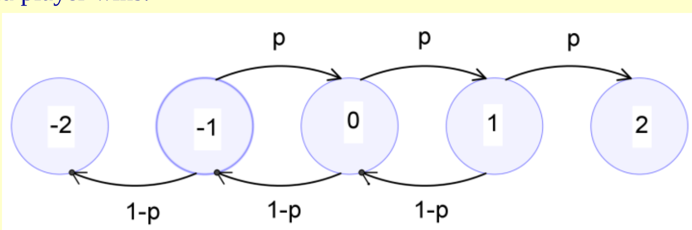



- How process evolves depends on the probability of moving from one state to another.
- The transition matrix $P$ is a matrix whose $(i,j)$th entry is transition probabilites $p_{ij}$ from $i$th state to $j$th state.

$$p_{i,i+1} = p$$
$$p_{i+1,i} = 1 - p$$

# Finite game

Modification: The chain will terminate at states **-2** or **2**– first player or the second player wins.




$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

# Tossing coin as a Markov chain

**Memoryless property**

The outcome of the *n*th toss is independent of the results of tosses 1,…, *n - 1*.
If we know what happened at time *n-1*, then any other information about the past does not affect the probability distribution for time n.

In general, a Markov chain is given by

- A state space $S$ – a countable set of states
- Transition probabilities $p_{ij}$
- Initial distribution $\alpha(0)$

# Finite game

*states* *S={-2 -1 0 1 2}*

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Initial distribution $\alpha(0) = (0,0,1,0,0)$

Prob. distribution after 1 game: $\alpha(1) = (0,1-p,0,p,0) = \alpha(0)\mathrm{P}$

Prob. distribution after 2 games: $\alpha(2) = \alpha(1)\mathrm{P} = \alpha(0)P^2$

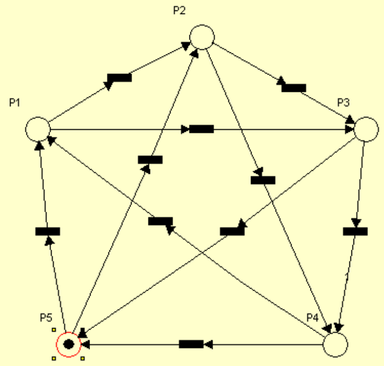
…

Prob. distribution after *n* games: $\alpha(n) = \alpha(n-1)\mathrm{P} = \alpha(0)P^n$

Probability distribution vector $\alpha(n)$ denotes the probabilities that the system is in each state at time *n*. Probability distribution vector $\alpha(n)$ depends upon the initial state of the system $\alpha(0)$ and transition matrix *P*.

# Transition matrix

square matric $P$, $p_{ij}$ means probability of transition form place P$i$ to place P$j$.




$$P = \begin{pmatrix} 0 & 0{,}5 & 0{,}5 & 0 & 0 \\ 0 & 0 & 0{,}5 & 0{,}5 & 0 \\ 0 & 0 & 0 & 0{,}5 & 0{,}5 \\ 0{,}5 & 0 & 0 & 0 & 0{,}5 \\ 0{,}5 & 0{,}5 & 0 & 0 & 0 \end{pmatrix}$$

$$a(1) = a(0)P$$
$$a(2) = a(1)P$$
$$\vdots$$
$$a(n+1) = a(n)P$$



$$P = \begin{pmatrix} 0 & 0{,}5 & 0{,}5 & 0 & 0 \\ 0 & 0 & 0{,}5 & 0{,}5 & 0 \\ 0 & 0 & 0 & 0{,}5 & 0{,}5 \\ 0{,}5 & 0 & 0 & 0 & 0{,}5 \\ 0{,}5 & 0{,}5 & 0 & 0 & 0 \end{pmatrix}$$

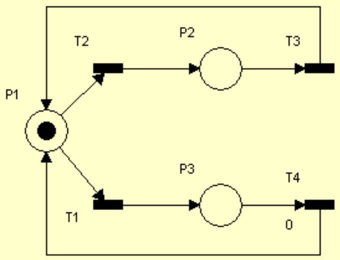
$$a(n+1) = a(n)P$$
$$\lim_{n\to\infty} a(n+1) = \lim_{n\to\infty} a(n) = a$$
$$a = aP$$

$$a = aP$$
$$a\left(P - E\right) = o$$
$$a\begin{pmatrix} P - E \left| \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \right. \end{pmatrix} = (0, \ldots, 0, 1)$$

```
clear;
P=[0,0.5,0.5,0,0;0,0,0.5,0.5,0;0,0,0,0.5,...
    0.5;0.5,0,0,0,0.5;0.5,0.5,0,0,0]
P^20
A=[(P-eye(5)),[1;1;1;1;1]]; %normalization
a=[0,0,0,0,0,1]/A  % A.a=[0,0,0,1]
```

# Example – Stochastic Petri Net

- State space = {P1, P2, P3}
- Transition matrix P
  - square matric $P$, $p_{ij}$ means probability of transition form place P$i$ to place P$j$.
  - Initial distribution $a(0)$ and transition matrix P uniquely determine the distribution $a(n)$ after n steps.




$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix};$$

$$a(0) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$a(1) \cdot P = a(0)$$

11

# Markov Chains

- A Markov Chain is a weighted digraph representing a discrete-time system that can be in any number of discrete states
- The transition matrix $P$ for a Markov chain is matrix of probabilities of moving from one state to another
- $p_{ij}$ = probability of moving from state i to j **is independent of what happened before** moving to state j and how one got to state i (Markov assumption)
- Sum of probabilities $\alpha(n)$ for each time $n$ must be one.

$$\alpha(n) = \alpha(n-1)\mathrm{P} = \alpha(0)P^n$$

# Birth-death chain

**Example 12.8.** *Birth-death chain.* This is a general model in which a population may change by at most 1 at each time step. Assume the size of a population is $x$. The *birth* probability $p_x$ is the transition probability to $x+1$, the *death* probability $q_x$ is the transition to $x-1$. and $r_x = 1 - p_x - q_x$ is the transition to $x$. Clearly, $q_0 = 0$. The transition matrix is now

$$\begin{bmatrix} r_0 & p_0 & 0 & 0 & 0 & \dots \\ q_1 & r_1 & p_1 & 0 & 0 & \dots \\ 0 & q_2 & r_2 & p_2 & 0 & \dots \\ & & & & & \ddots \end{bmatrix}$$

# Stable (regular) process

- A common question arising in Markov-chain models is, what is the long-term probability that the system will be in each state?
- The vector containing these long-term probabilities is called the **steady-state vector** of the Markov chain

- Stable process – $\alpha(n)$ tends to a limit $\alpha$ as $n \to \infty$, this steady state limit does not depend on the initial state.
- Key to the study of Markov chains is the study of powers of transition matrix $P$.

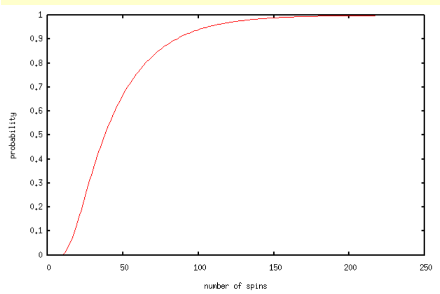
- Any nonnegative vector which satisfies $\alpha = \alpha$ P and whose components sum to one is called a **stationary** probability distribution of the Markov Chain.

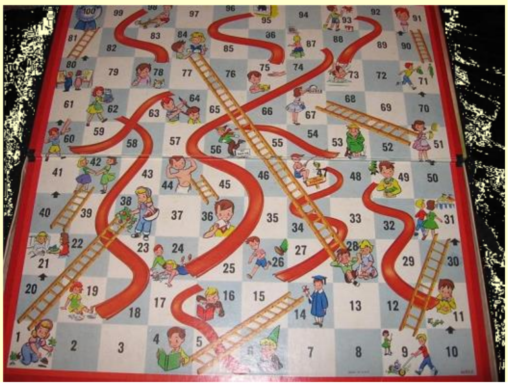

# How the Game is Played

- Chutes and Ladders is a board game where players spin a pointer to determine how they will advance
- The board consists of 100 numbered squares
- The objective is to land on square 100
- However, the board is filled with chutes and ladders, which move a player backward or forward if landed on.

# Simulation

- Objectives:
  - Find frequencies for being at each position
  - Find mean number of moves to win
  - Find standard deviation
  - Simulate a large number of games



# Results of Transition Matrix

- After 1000 moves, the probability vector reached a limit of {0,0,…1}
- This means that after 1000 moves, the game is expected to be won!!!

# Results of simulation

- We ran 250,000 games
- Mean is approximately 39.65 moves to reach square 100.
- The standard deviation is approximately 24.00

# DTMC simulation

DTMC with 3 states is given by transition matrix P and initial state $\alpha(1) = (1,0,0)$.

1. Simulate one run with 100 steps. (resp. 500 , 1000 steps)
2. Estimate the steady-state distribution

$$P = \begin{pmatrix} 0{,}2 & 0{,}3 & 0{,}5 \\ 0{,}1 & 0{,}1 & 0{,}8 \\ 0{,}4 & 0{,}3 & 0{,}3 \end{pmatrix}$$

```
P=[0.2,0.3,0.5;0.1,0.1,0.8;0.4,0.3,0.3];
steps=100;
states=zeros(steps,1)
states(1)=1% start,

for i=1:steps-1
  states(i+1)=find(rand<cumsum(P(states(i),:)),1)
end

%steady-state distribution
freq=hist(states,1:3);
a_est=freq/steps
```

DTMC.m

# DTMC analytical solution

DTMC with 3 states is given by transition matrix P and initial state $\alpha(1) = (1,0,0)$.

1. Determine $P^n$
2. Determine the steady-state distribution

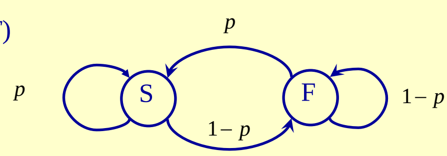
$$P = \begin{pmatrix} 0,2 & 0,3 & 0,5 \\ 0,1 & 0,1 & 0,8 \\ 0,4 & 0,3 & 0,3 \end{pmatrix}$$

```
%Steade-state distribution lim a(n)=a(0)P^n ever since P^8
% a = (0.2708, 0.2500, 0.4792)
limP=P^10
A=[(P-eye(length(P)))';[1,1,1]]
a=A\[0;0;0;1]  % Aa=[0,0,0,1]' together with normalize
```

## Bernoulli trials as Markov chain

There are only **two possible outcomes** for each trial, often designated success or failure.
The probability of success, p, is the same for every trial.

State space = $(S, F)$




$$P = \begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$

Probability vector a(n) - probability of states *S* or *F* after n trials.

$$\left(a_S(1), a_F(1)\right) = \left(p(a_S(0)+a_F(0)), (1-p)(a_S(0)+a_F(0))\right) = (p, 1-p)$$

$$a = aP$$
$$\left(a_S, a_F\right) = \left(p(a_S + a_F), (1-p)(a_S + a_F)\right)$$
$$\left(a_S, a_F\right) = (p, 1-p)$$

Bernoulli.m

## Geometric distribution as absorbing Markov chain

The geometric distribution is the only discrete **memoryless** random distribution. It is a discrete analog of the exponential distribution.

The probability distribution of the number X of Bernoulli trials needed to get one success

```
%an2 = an(2) Probability of failure
state after n steps
for n = 1:15;
an=a0*P^n
an2(n)=an(2)
end
plot(1:15,an2,'-*')
```

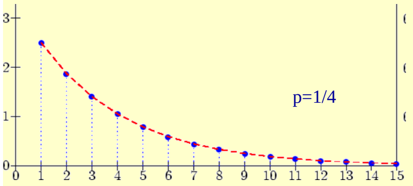
$$P = \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix};\ a(0) = (0,1)$$

$$a(1) = (p, 1-p)$$

$$a(2) = (2p - p^2, (1-p)^2)$$

$$a(n) = (1-(1-p)^n, (1-p)^n)$$

An absorbing state is a state that, once entered, cannot be left.




$$a = aP$$

$$(a_S, a_F) = (a_S + pa_F, (1-p)a_F)$$

$$(a_S, a_F) = (1, 0)$$

## Geometric distribution (control and security measures)

Example: Functioning of some device is inspected once a day – at 6p.m. Probablity $p$ of defect (success!) is same for the whole of observed time: $p = 0.1$.

Estimate the probabilty that errorless period is longer then 5 days.

Estimate the average errorless period.

```
p=0.1;n=500;
days=zeros(n,1);
for i = 1:n
    while rand > p %failure (OK)
    days(i)=days(i)+1;
    end
end
mean(days))
sum(days>5)/n
```

$$P = \begin{pmatrix} 1 & 0 \\ p & 1-p \end{pmatrix};\ a(0) = (0,1)$$

$$a(n) = (1-(1-p)^n, (1-p)^n)$$

$$p = 0.1$$

$$a(6) = (1-0.9^6;\ 0.9^6)$$

$$1 - cdf(6) = (1-p)^6 = 0.53$$

$$E[X] = \frac{1-p}{p} = 9(days)$$