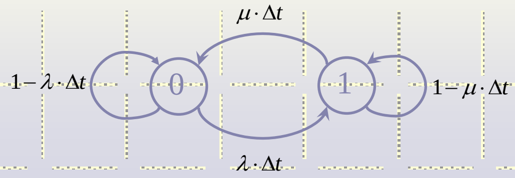
1 service station, empty buffer
M / M / 1/0

Differential transitions



M / M / 1/0
Steady state distribution

```
lambda = 1/2; % inter-arrival time = 1/lambda [s]
mu=1/2;       % inverse mean service time
T=60*60;      % time of observation

Q=zeros(ceil(T/0.1),1); % number of customers
    for i=1:ceil(T/0.1)
        dQ=0;
        if rand < lambda*0.1 && Q(i)==0
          dQ=1;
        end
        if rand < mu*0.1  && Q(i)==1
          dQ=-1;
        end
       Q(i+1)=Q(i)+dQ;
    end
p0=sum(Q==0)/length(Q);
```

Proportion of rates $\rho = \frac{\lambda}{\mu}$

$$p = \left(\frac{\mu}{\lambda + \mu}, \frac{\lambda}{\lambda + \mu} \right) = \left(\frac{1}{1+\rho}, \frac{\rho}{1+\rho} \right)$$

M / M / 1/0

Steady state distribution

Proportion of rates $\rho = \frac{\lambda}{\mu}$

$$p = \left(\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu}\right) = \left(\frac{1}{1+\rho}, \frac{\rho}{1+\rho}\right)$$

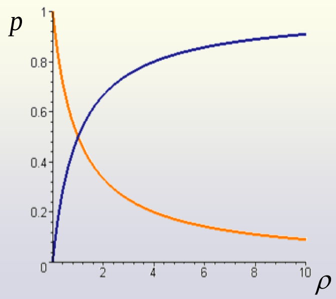
$$p = \left(\frac{\mu}{\lambda+\mu}, \frac{\lambda}{\lambda+\mu}\right)$$

Probability of empty system

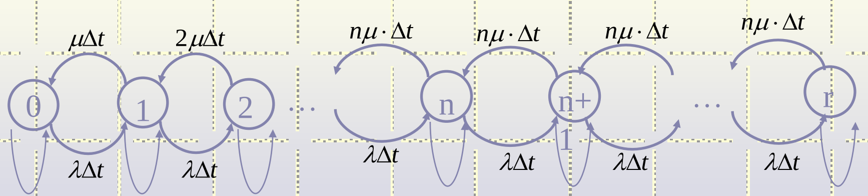
Probability of full system

- X – number of customers {0,1}
- Expected value: $E[X] = p_1 = \frac{\rho}{1+\rho}$

	probability
Rejection of a customer:	p_1
Acceptance of a customer:	p_0
Utilization E[S]	$p_1 \cdot 100\%$



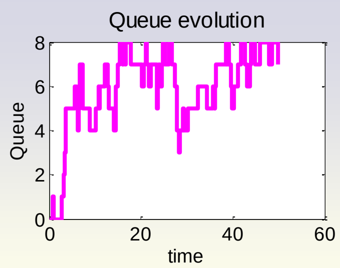
Markovian Queues
(n service stations, r - limit for bufer)
M / M / n / r



M / M / n / r

```
function[Q,t,p_lost]=MMnr(lambda, mu,n,r)
%r capacity of buffer - max. queue length
dt=0.1; T=60*60;
L=0;S=0;        %S-served, L-lost
na=0;           %active servers
Q=zeros(ceil(T/dt),1);
        for i=1:ceil(T/dt)
            dQ=0;
            if rand < lambda*dt
                if Q(i)<n+r
                na=min(na+1,n);dQ=dQ+1; S=S+1;
                else
                 L=L+1;
                end
            end
            if rand < na*mu*dt && Q(i)>0
                dQ=dQ-1;na=min(Q(i)-1,n)
            end
             Q(i+1)=Q(i)+dQ;
        end
        p_lost=L/(L+S);
t=0:dt:T;
```

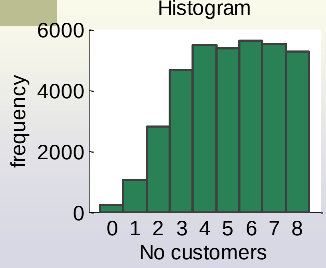
M / M / n / r




```
lambda=1;mu=1/5;n=5;r=3;   %(E[X] = 5.2)

[Q,t]=MMnr(lambda,mu,n,r);
stairs(t(1:500),Q (1:500),'m','Linewidth',3)
```

- **Investigate the evolution of the Queue length**
- **Estimate the mean number of customers**
- **Estimate acceptance of a customer**
- **Draw the histogram of the queue lengths.**

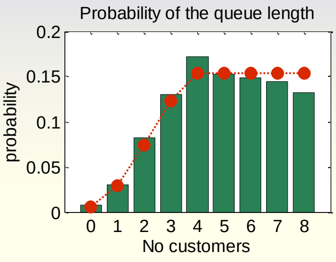


Theoretical probability of full queue
p_{n+r} = 0,151

M / M / n / r
Theory

$$0 \le k \le n; \quad p_{k+1} = \frac{\lambda p_k}{\mu(k+1)}, \Rightarrow \quad p_k = \left(\frac{\lambda}{\mu}\right)^k \frac{p_0}{k!}$$

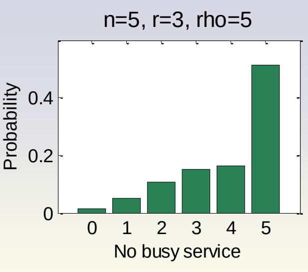
$$n \le k \le n+r; \quad p_k = \left(\frac{\lambda}{n\mu}\right)^{k-n} p_n, \Rightarrow \quad p_k = \frac{p_0}{n!\,n^{k-n}}\left(\frac{\lambda}{\mu}\right)^k$$



Compare relative frequency with theoretical formula

```
lambda=1;mu=1/5;n=5;r=3;
[prob_k, EX] = MMnr_formulae(lambda, mu, n, r);
hold on
k=0:n+r;
plot(k,prob_k,'r--','LineWidth',3)
hold off
```

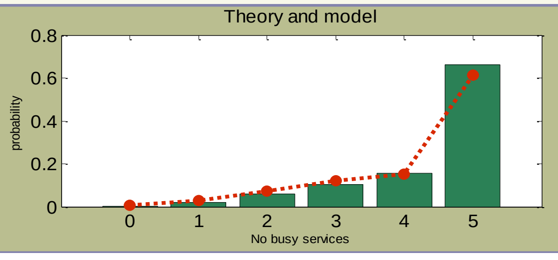
M / M / n / r



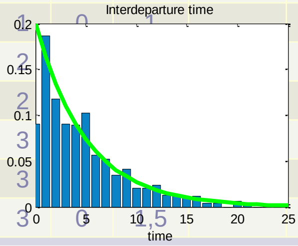

```
lambda=1;mu=1/5;n=5;r=3;
[Q,t]=MMnr(lambda,mu,n,r);
```

Draw the histogram of busy services
Estimate the utilization of the service stations.

```
k=0:n+r;
prob=hist(Q,k)/length(Q); % empirical distribution
probS=[prob(1:n),sum(prob(n+1:n+r+1))]
bar(0:n,probS)
ES=sum(probS.*[0:n]) %Expected value- No busy service
Utilization = ES/n;
```



Q	dQ	t	times_ input	times_ output
0	0	0	0,3	0,5
0	0	0,1	0,8	
0	0	0,2	1,1	
1	1	0,3	1,3	
1	0	0,4		
0	-1	0,5		
0	0	0,6		
0	0	0,7		
1	1	0,8		
1	0	0,9		
1	0	1		
2	[illegible]	[illegible]	[illegible]	
2	[illegible]	[illegible]	[illegible]	
3	[illegible]	[illegible]	[illegible]	
3	[illegible]	[illegible]	[illegible]	
3	0	1,5	[illegible]	



M / M / n / r

`times_input`: Moments of arrivals

`times_output`: Moments of departures– end of service

```
dQ=diff(Q);
times_input=t(dQ==1); % čas
times_output=t(dQ==-1);
```

Burke’s output theorem

```
hist(diff(times_output),20);
% exponential distr.
mean(diff(times_output))
```

M / M / n / r - waiting time

W_i – time i-th customer spends in system (queue+service)

```
W=times_output-
times_input(1:length(times_output));

fprintf('Mean W %6.2g \n',mean(W))
fprintf('Mean Q %6.2g \n',mean(Q))
```

times_ input	times_ output	W
0,3	0,5	0,2
0,8	1,2	0,4
1,1	1,7	0,6
1,3	1,9	0,6
2,1	2,5	0,4
2,8	4,1	1,3
2,7	4,5	1,8
3,2	4,9	1,7
4,1		
4,3		

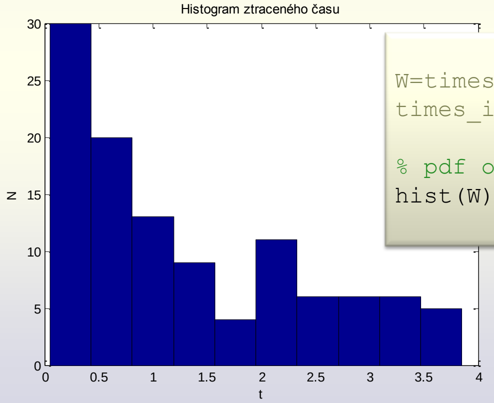
Little’s Law

Hold for any G/G/n system with arbitrary service discipline

$$E[Q] = E[W] * \lambda$$

Probability distribution of waiting time W

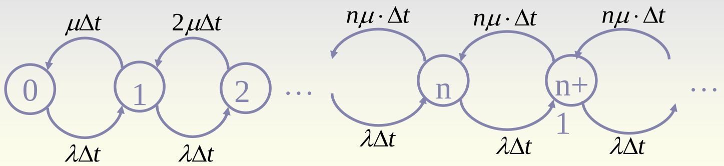
Distribution function



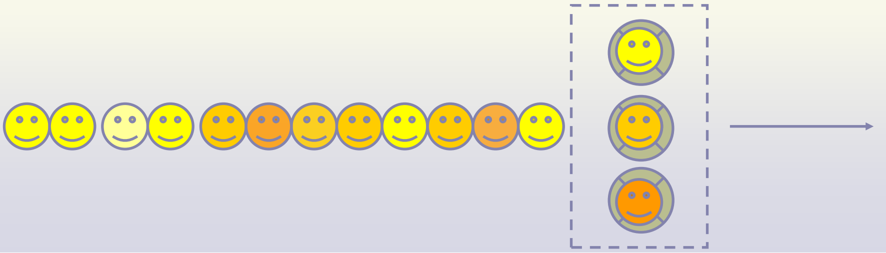

```
W=times_output-
times_input(1:length(times_output));

% pdf of waiting time
hist(W);figure(gcf);
```

M /M / n / ∞
n service stations, Infinitely long queue



Differential transitions	**probability**
Completion of service for *k* busy stations	*k*.μΔt +o(Δt).
Arrival of a new customer	λ.Δt +o(Δt).



M /M / n / ∞

$$0 \le k \le n; \quad p_k = \frac{\lambda p_{k-1}}{\mu k}, \Rightarrow p_k = \left(\frac{\lambda}{\mu}\right)^k \frac{p_0}{k!}$$

$$n \le k; \quad p_k = \left(\frac{\lambda}{n\mu}\right)^{k-n} p_n, \Rightarrow p_k = \frac{p_0}{n! n^{k-n}} \left(\frac{\lambda}{\mu}\right)^k$$

$$p_0 = \left(\sum_{k=0}^{n-1} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} + \frac{n^n}{n!} \sum_{k=n}^{\infty} \frac{1}{n^k} \left(\frac{\lambda}{\mu}\right)^k \right)^{-1} = \left(\sum_{k=0}^{n-1} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!} + \frac{n\left(\frac{\lambda}{\mu}\right)^n}{n!\left(n - \frac{\lambda}{\mu}\right)} \right)^{-1}$$

Stability condition: $\rho_n = \frac{\lambda}{n\mu} < 1 \quad \Rightarrow \quad \lambda < n\mu$

Performance Measures M /M / n / ∞

$$k \le n; \; p_k = \rho^k \frac{p_0}{k!} \qquad p_0 = \left(\sum_{k=0}^{n-1} \frac{\rho^k}{k!} + \frac{n^n}{n!} \sum_{k=n}^{\infty} \left(\frac{\rho}{n}\right)^k \right)^{-1}$$

$$k \ge n; \; p_k = \frac{p_0}{n! n^{k-n}} \rho^k$$

- Mean (expected value) busy service stations

$$E[S] = \sum_{k=0}^{n} k p_k + \sum_{k=n+1}^{\infty} n p_k = p_0 \sum_{k=1}^{n} k \frac{\rho^k}{k!} + p_0 \sum_{k=n+1}^{\infty} n \frac{\rho^k n^{n-k}}{n!} = p_0 \rho \left[\sum_{k=0}^{n-1} \frac{\rho^k}{k!} + \frac{n^n}{n!} \sum_{k=n}^{\infty} \left(\frac{\rho}{n}\right)^k \right] = \rho = \frac{\lambda}{\mu}$$

- Mean queue length

$$E[F] = \sum_{k=0}^{\infty} k p_{n+k} = \sum_{k=0}^{\infty} k \left(\frac{\rho}{n}\right)^k p_n = p_n \sum_{k=0}^{\infty} k \rho_n^{\,k} = \frac{p_n \rho_n}{(1-\rho_n)^2} \qquad kde \quad \rho_n = \frac{\lambda}{n\mu}$$

- Mean number of customers (served + queue)

$$E[X] = \sum_{k=0}^{\infty} k p_k = \sum_{k=1}^{n} k p_k + \sum_{k=n+1}^{\infty} k p_k = \sum_{k=1}^{n} k p_k + \sum_{k=n+1}^{\infty} n p_k + \sum_{k=n+1}^{\infty} (k-n) p_k = E[S] + E[F]$$

- Utilization

$$\frac{E[S]}{n} = \frac{\rho}{n} = \frac{\lambda}{n\mu} = \rho_n$$

Exercise

- In a gas station there are **4** gas pumps. Cars arrive at the gas station according to a Poisson process. The arrival rate is **20** cars per hour. The places for waiting are limited to having a maximum of **3** cars. Cars are served in order of arrival. The service time (i.e. the time needed for pumping and paying) is exponential. The mean service time is **10** minutes.

- Determine the stationary distribution of the number of cars at the gas station.
- Determine the mean number of cars at the gas station.
- Determine the mean sojourn time (waiting time plus service time) of cars.