

# TABLE OF CONTENTS

<b>Preface</b> .....	<b>1</b>
<b>1. Introduction</b> .....	<b>3</b>
1.1 Ordinary differential equations in the 19th century .....	3
1.2 Motivation to the definition of product integral .....	7
1.3 Product integration in physics .....	9
1.4 Product integration in probability theory .....	9
<b>2. The origins of product integration</b> .....	<b>13</b>
2.1 Product integration in the work of Vito Volterra .....	15
2.2 Basic results of matrix theory .....	16
2.3 Derivative of a matrix function .....	19
2.4 Product integral of a matrix function .....	22
2.5 Continuous matrix functions .....	32
2.6 Multivariable calculus .....	39
2.7 Product integration in complex domain .....	51
2.8 Linear differential equations at a singular point .....	59
<b>3. Lebesgue product integration</b> .....	<b>65</b>
3.1 Riemann integrable matrix functions .....	66
3.2 Matrix exponential function .....	71
3.3 The indefinite product integral .....	74
3.4 Product integral inequalities .....	76
3.5 Lebesgue product integral .....	79
3.6 Properties of Lebesgue product integral .....	83
3.7 Double and contour product integrals .....	91
3.8 Generalization of Schlesinger's definition .....	95
<b>4. Operator-valued functions</b> .....	<b>99</b>
4.1 Integral operators .....	100
4.2 Product integral of an operator-valued function .....	102
4.3 General definition of product integral .....	108
<b>5. Product integration in Banach algebras</b> .....	<b>111</b>
5.1 Riemann-Graves integral .....	112
5.2 Definition of product integral .....	114
5.3 Useful inequalities .....	116
5.4 Properties of product integral .....	118

5.5 Integrable and product integrable functions .....	122
5.6 Additional properties of product integral .....	127
<b>6. Kurzweil and McShane product integrals .....</b>	<b>131</b>
6.1 Kurzweil and McShane integrals .....	132
6.2 Product integrals and their properties .....	133
<b>7. Complements .....</b>	<b>139</b>
7.1 Variation of constants .....	139
7.2 Equivalent definitions of product integral .....	140
7.3 Riemann-Stieltjes product integral .....	142
<b>Bibliography .....</b>	<b>145</b>