Vector spaces

1) Exercise : decide whether the vectors are linearly dependent

a) $v = (2,1,3,1)$, $u = (1,2,0,1)$, $w = (-1,1,-3,0)$

$$\begin{array}{c} u \\ v \\ w \end{array}\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 3 & 1 \\ -1 & 1 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 3 & -1 \\ 0 & 3 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -3 & 3 & -1 \\ \boxed{0 \;\; 0 \;\; 0 \;\; 0} \end{pmatrix} \Longrightarrow$$

a zero vector

the vectors u, v, w are linearly dependent

b) $v = (2,3,-5)$, $u = (1,-1,1)$, $w = (3,2,-2)$

$$\begin{array}{c} u \\ v \\ w \end{array}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & -5 \\ 3 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 5 & -7 \\ 0 & 5 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 5 & -7 \\ 0 & 0 & 2 \end{pmatrix} \Longrightarrow$$

the stairstep matrix [3 nonzero vectors]

the vectors u, v, w are linearly independent

c) $v = (1,0,3)$, $u = (-3,0,-9)$, $w = (1,1,2)$

we see $u = -3 \cdot v = -3 \cdot (1,0,3) = (-3,0,-9)$

$\longrightarrow$ the vectors u, v, w are linearly dependent

$$\begin{array}{c} v \\ u \\ w \end{array}\begin{pmatrix} 1 & 0 & 3 \\ -3 & 0 & -9 \\ 1 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ \boxed{0 \;\; 0 \;\; 0} \\ 0 & 1 & -1 \end{pmatrix} \Longrightarrow \quad -\text{“}-$$

a zero vector

d) $v = (3,4,3)$ $u = (1,3,-1)$, $w = (1,-1,1)$ $\Longrightarrow$ the vectors u, w, v are linearly independent

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 3 & -1 \\ 3 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 7 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & -14 \end{pmatrix}$$

the stairstep matrix [3 nonzero vectors]

e) $v=(1,2,0,0),\ u=(0,1,1,0),\ w=(1,0,0,1),\ q=(1,1,-1,1)$

$$\begin{array}{c} v\\ u\\ w\\ q \end{array}\begin{pmatrix} 1&2&0&0\\ 0&1&1&0\\ 1&0&0&1\\ 1&1&-1&1 \end{pmatrix} \sim \begin{pmatrix} 1&2&0&0\\ 0&1&1&0\\ 0&-2&0&1\\ 0&-1&-1&1 \end{pmatrix} \sim \begin{pmatrix} 1&2&0&0\\ 0&1&1&0\\ 0&0&2&1\\ 0&0&0&1 \end{pmatrix} \Longrightarrow$$

the vectors v, u, w, q are <u>linearly independent</u>

f) $v=(4,7,1,0),\ u=(2,3,-1,2),\ w=(1,2,1,-1),\ q=(5,7,-4,7)$

$$\begin{array}{c} w\\ u\\ v\\ q \end{array}\begin{pmatrix} 1&2&1&-1\\ 2&3&-1&2\\ 4&7&1&0\\ 5&7&-4&7 \end{pmatrix} \sim \begin{pmatrix} 1&2&1&-1\\ 0&-1&-3&4\\ 0&-1&-3&4\\ 0&-3&-9&12 \end{pmatrix} \sim \begin{pmatrix} 1&2&1&-1\\ 0&-1&-3&4\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \Longrightarrow$$

two zero vectors

the vectors v, u, w, q are <u>linearly dependent</u>

g) $v=(2,1,1,1),\ u=(1,2,-1,2),\ w=(1,-1,-1,1),\ q=(1,2,2,-2)$

$$\begin{array}{c} u\\ v\\ w\\ q \end{array}\begin{pmatrix} 1&2&-1&2\\ 2&1&1&1\\ 1&1&-1&1\\ 1&2&2&-2 \end{pmatrix} \sim \begin{pmatrix} 1&2&-1&2\\ 0&-3&3&-3\\ 0&-1&0&-1\\ 0&0&3&-4 \end{pmatrix} \sim \begin{pmatrix} 1&2&-1&2\\ 0&-3&3&-3\\ 0&0&3&0\\ 0&0&3&-4 \end{pmatrix} \begin{array}{c} \cdot(-\frac{1}{3})\\ \cdot(-\frac{1}{3})\\ \cdot(-1) \end{array} \sim \begin{pmatrix} 1&2&-1&2\\ 0&1&-1&1\\ 0&0&1&0\\ 0&0&0&4 \end{pmatrix}$$

$\Longrightarrow$ the vectors u, v, w, q are linearly <u>independent</u>

h) $p=(2,1,3,-1)\quad q=(-1,1,-3,1),\ r=(4,5,3,-1),\ s=(1,5,-3,1)$

$$\begin{array}{c} q\\ p\\ r\\ s \end{array}\begin{pmatrix} -1&1&-3&1\\ 2&1&3&-1\\ 4&5&3&-1\\ 1&5&-3&1 \end{pmatrix} \sim \begin{pmatrix} -1&1&-3&1\\ 0&3&-3&1\\ 0&9&-9&3\\ 0&6&-6&2 \end{pmatrix} \sim \begin{pmatrix} -1&1&-3&1\\ 0&3&-3&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$$

$\Longrightarrow$ the vectors p, q, r, s are <u>linearly</u> <u>dependent</u>

ch) $p=(2,1,1),\ q=(0,1,-1),\ r=(2,-1,1),\ s=(1,0,1)$

4x tree taples $\Longrightarrow$ the vectors p, q, r, s are linearly <u>dependent</u>

$$\begin{array}{c} s\\ q\\ p\\ r \end{array}\begin{pmatrix} 1&0&1\\ 0&1&-1\\ 2&1&1\\ 2&-1&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&1\\ 0&1&-1\\ 0&1&-1\\ 0&-1&3 \end{pmatrix} \sim \begin{pmatrix} 1&0&1\\ 0&1&1\\ 0&0&0\\ 0&0&2 \end{pmatrix}$$

two same vectors

2) Exercise: Let $a \in \mathbb{R}$ be a parameter. Decide when the vectors are linearly dependent and when independent.

a) $v = (a, -2, 1), \quad u = (3, 2a, -1), \quad w = (a^2, 1, a-1)$

$$\begin{array}{c} u \\ v \\ w \end{array}\begin{pmatrix} 3 & 2a & -1 \\ a & -2 & 1 \\ a^2 & 1 & a-1 \end{pmatrix} \begin{array}{c} -\frac{a}{3})+ \\ \sim \; -a \\ +) \end{array} \begin{pmatrix} 3 & 2a & -1 \\ 0 & -2a^2-6 & 3+a \\ 0 & 2a+1 & -1 \end{pmatrix} \begin{array}{c} -1 \;\; 2a+1 \\ \sim \;\;)+ \\ 2a^2+6 \end{array} \begin{pmatrix} 3 & 2a & -1 \\ 0 & 2a^2+6 & -3-a \\ 0 & 0 & 7a-3 \end{pmatrix}$$

3 is not 0 for each $a \in \mathbb{R}$

$-2a^2-6 \neq 0$ for each $a \in \mathbb{R}$

$7a-3=0$

$a = \frac{3}{7}$

I. $a = \frac{3}{7}$

$$\begin{pmatrix} 3 & \frac{6}{7} & -1 \\ 0 & 6+\frac{18}{49} & -\frac{24}{7} \\ 0 & 0 & 0 \end{pmatrix}$$

the vectors v, w, u are linearly dependent

II. $a \neq \frac{3}{7}$

$$\begin{pmatrix} 3 & \cdot & \cdot \\ 0 & ☺ & \cdot \\ 0 & 0 & \triangle \end{pmatrix}$$

☺ an $\triangle$ are non zero

$\Longrightarrow$ the vectors u, v, w are linearly independent

b) $v = (a, -4, -1), \quad u = (4, -6, -3), \quad (1, 1, -a)$

$$\begin{pmatrix} 4 & -6 & -3 \\ 1 & 1 & -a \\ a & -4 & -1 \end{pmatrix} \begin{array}{c})+ \\ -4 \sim \; -a \\ +) \end{array} \begin{pmatrix} 4 & -6 & -3 \\ 0 & -10 & 4a-3 \\ 0 & -4-a & a^2-1 \end{pmatrix} \begin{array}{c} 4+a \\ \sim \;\;)+ \\ -10 \end{array} \begin{pmatrix} 4 & -6 & -3 \\ 0 & -10 & 4a-3 \\ 0 & 0 & -6a^2+13a-2 \end{pmatrix}$$

4 is not 0 for each $a \in \mathbb{R}$

$$-6a^2+13a-2 = 0$$

$$a_{1,2} = \frac{-13 \pm \sqrt{169-48}}{-12} = -\frac{13 \pm 11}{12} = \left\langle \begin{array}{l} 2 \\ \frac{1}{6} \end{array} \right.$$

I. $a = 2$

$$\begin{pmatrix} 4 & -6 & -3 \\ 0 & -10 & 9 \\ \boxed{0 \quad 0 \quad 0} \end{pmatrix} \Longrightarrow$$ the vectors u, v, w are linearly dependent

II. $a = \frac{1}{6}$

$$\begin{pmatrix} 4 & -6 & -3 \\ 0 & -10 & -\frac{7}{3} \\ \boxed{0 \quad 0 \quad 0} \end{pmatrix} \Longrightarrow$$ the vectors u, v, w are linearly dependent

III. $a \neq 2 \wedge a \neq \frac{1}{6}$

$$\begin{pmatrix} 4 & \cdot & \cdot \\ 0 & -10 & \cdot \\ 0 & 0 & \triangle \end{pmatrix} \Longrightarrow$$ $\triangle$ is non zero
the vectors u, v, w are linearly independent

c) $v = (2, 3, -1)$, $u = (a, 4, 2)$

$$\begin{pmatrix} 2 & 3 & -1 \\ a & 4 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 3 & 1 \\ 0 & -8+3a & a-4 \end{pmatrix}$$

$-8+3a = 0$ and $a-4 = 0$

$a = \frac{8}{3}$ and $a = 4$ it is not OK

$\longrightarrow$ the vectors u, v are linearly independent for each $a \in \mathbb{R}$

d) $v = (1,1,1)$, $u = (1, a, 1)$, $w = (2, 2, a)$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 2 & 2 & a \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & a-1 & 0 \\ 0 & 0 & a-2 \end{pmatrix} \longrightarrow$$ $a-1=0$ or
$a=1$

$\longrightarrow$ $a-2=0$
$a=2$

I. $a=1$

$$\begin{pmatrix} 1 & 1 & 1 \\ \boxed{0 \quad 0 \quad 0} \\ 0 & 0 & -1 \end{pmatrix} \Longrightarrow$$ the vectors v, u, w are linearly dependent

II. $a=2$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} \end{pmatrix} \longrightarrow$$ the vectors v, u, w are linearly dependent

III. $a \neq 1 \wedge a \neq 2$

Δ, ☼ are nonzero

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \Delta & \cdot \\ 0 & 0 & ☼ \end{pmatrix} \Longrightarrow$$ the vectors v, u, w are linearly independent

c) is the same as the b

3) Exercise: Find a basis of vector space V spanned by next vectors. Calculate its dimension.

a) $u_1 = (1,2,3,4)$, $u_2 = (1,5,1,2)$, $u_3 = (1,1,2,3)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 5 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & -2 & -2 \\ 0 & 0 & -5 & -5 \end{pmatrix} \Longrightarrow$$ the vectors u_1, u_2, u_3 are linearly independent

3 LI vectors

$$B = \{ \overset{u_1}{(1,2,3,4)}, \overset{u_2}{(1,5,1,2)}, \overset{u_3}{(1,1,2,3)} \} \qquad \dim V = 3$$

3 vectors

b) $u_1 = (1,2,3,2)$, $u_2 = (0,1,1,0)$, $u_3 = (1,0,1,2)$

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ the vectors u_1, u_2, u_3 are LD!

$\Longrightarrow$ the vectors u_1 and u_2 are LI

2 LI vectors

$$B = \{ \overset{u_1}{(1,2,3,2)}, \overset{u_2}{(0,1,1,0)} \} \qquad \dim V = 2$$

2 vectors

c) $u_1 = (1,2,3,2),\ u_2 = (0,1,-1,3),\ u_3 = (1,2,1,6),\ u_4 = (2,4,3,5)$

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & 3 \\ 1 & 2 & 1 & 6 \\ 2 & 4 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

the vectors u_1, u_2, u_3, u_4 are linearly independent

4 LI vectors

$B = \{ \overset{u_1}{(1,2,3,2)}, \overset{u_2}{(0,1,-1,3)}, \overset{u_3}{(1,2,1,6)}, \overset{u_4}{(2,4,3,5)} \}$ $\quad$ $\dim V = 4$

4 LI vectors

d) $u_1 = (3,1,5,4),\ u_2 = (2,2,3,3),\ u_3 = (1,-1,2,1),\ u_4 = (1,3,1,2)$

$$\begin{pmatrix} 3 & 1 & 5 & 4 \\ 2 & 2 & 3 & 3 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 5 & 4 \\ 0 & 4 & -1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & -8 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 5 & 4 \\ 0 & 4 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the vectors u_1, u_2, u_3, u_4 are linearly dependent

$\Downarrow$ the vectors u_1 and u_2 are linearly independent

2 LI vectors

$B = \{ \overset{u_1}{(3,1,5,4)}, \overset{u_2}{(2,2,3,3)} \}$ $\quad$ $\dim V = 2$

2 LI vectors

e) $u_1 = (1,0,2,-3),\ u_2 = (3,2,1,-5),\ u_3 = (-1,2,1,-2),\ u_4 = (-3,0,2,0)$

$$\begin{pmatrix} 1 & 0 & 2 & -3 \\ 3 & 2 & 1 & -5 \\ -1 & 2 & 1 & -2 \\ -3 & 0 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -5 & 4 \\ 0 & 2 & 3 & -5 \\ 0 & 0 & 8 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 8 & -9 \\ 0 & 0 & 8 & -9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 8 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 LI vectors

the vectors u_1, u_2, u_3, u_4 are linearly dependent

$\Downarrow$ the vectors u_1, u_2, u_3 are linearly independent

$B = \{ \overset{u_1}{(1,0,2,-3)}, \overset{u_2}{(3,2,1,-5)}, \overset{u_3}{(-1,2,1,-2)}, (-3,0,2,0) \}$ $\quad$ $\dim V = 3$

3 LI vectors

f) $u_1 = (3,1,5,4),\ u_2 = (2,2,3,3),\ u_3 = (1,-1,2,1),\ u_4 = (1,3,1,2)$

$$\begin{pmatrix} 3 & 1 & 5 & 4 \\ 2 & 2 & 3 & 3 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 5 & 4 \\ 0 & 4 & -1 & 1 \\ 0 & 4 & -1 & 1 \\ 0 & 8 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 5 & 4 \\ 0 & 4 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2 LI vectors

$B = \{(3,1,5,4),\ (2,2,3,3)\}$, $\dim V = 2$

g) $u_1 = (3,-1,-3,2),\ u_2 = (1,2,0,-3),\ u_3 = (1,2,1,2),\ u_4 = (5,1,-3,2)$

$$\begin{pmatrix} 3 & -1 & -3 & 2 \\ 1 & 2 & 0 & -3 \\ 1 & 2 & 1 & 2 \\ 5 & 1 & -3 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & -3 & 2 \\ 0 & -7 & -3 & 11 \\ 0 & -7 & -6 & -4 \\ 0 & -9 & -3 & 17 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & -3 & 2 \\ 0 & -7 & -3 & 11 \\ 0 & 0 & -3 & -15 \\ 0 & 0 & 6 & 20 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 & -3 & 2 \\ 0 & -7 & -3 & 11 \\ 0 & 0 & -3 & -15 \\ 0 & 0 & 0 & -10 \end{pmatrix}$$

4 LI vectors

$B = \{(3,-1,-3,2),\ (1,2,0,-3),\ (1,2,1,2),\ (5,1,-3,2)\}$, $\dim V = 4$

h) $u_1 = (5,7,-1,3),\ u_2 = (1,-3,8,2),\ u_3 = (9,17,-10,4),\ u_4 = (-2,6,-16,-4)$

$$\begin{pmatrix} 5 & 7 & -1 & 3 \\ 1 & -3 & 8 & 2 \\ 9 & 17 & -10 & 4 \\ -2 & 6 & -16 & -4 \end{pmatrix} \sim \begin{pmatrix} 5 & 7 & -1 & 3 \\ 0 & 22 & -41 & -7 \\ 0 & 35 & -82 & -14 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 5 & 7 & -1 & 3 \\ 0 & 22 & -41 & -7 \\ 0 & 0 & 369 & 63 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3 LI vectors

$B = \{(5,7,-1,3),\ (1,-3,8,2),\ (9,17,-10,4)\}$, $\dim V = 3$

4) Exercise: Find a basis of the vector space spaned by the set $S = \{u_1, u_2, u_3\}$ which contains the vector v.

a) $u_1 = (0,1,-3,4),\ u_2 = (2,2,2,2),\ u_3 = (1,-1,3,7);\ v = (1,4,-4,-1)$

$$\begin{array}{c} v \\ S\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array} \begin{pmatrix} 1 & 4 & -4 & -1 \\ 0 & 1 & -3 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & -1 & 3 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -4 & -1 \\ 0 & 1 & -3 & 4 \\ 0 & -6 & 10 & 4 \\ 0 & -5 & 7 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -4 & -1 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -8 & 28 \\ 0 & 0 & -8 & 28 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -4 & -1 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & -8 & 28 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} v \\ u_1 \\ \text{"}u_2\text{"} \\ \\ \end{array}$$

3 LI vectors

$\Longrightarrow$ the vectors u_1, u_2, u_3 a v are linearly dependent $\Longrightarrow$ the vector v can be expressed as the linear combination of the vectors u_1, u_2, u_3

$\Longrightarrow$ the exercise <u>has a solution</u>

$$B = \{ \underbrace{v, u_1, u_2}_{\text{there must be three vectors}} \} = \{(1,4,-4,-1), (0,1,-3,4), (2,2,2,2)\}$$

b) $u_1 = (1,2,2,-2)$, $u_2 = (1,2,-1,2)$, $u_3 = (2,1,1,1)$; $v = (1,-1,-1,-1)$

$$\begin{array}{c} v \\ S\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array} \begin{pmatrix} 1 & -1 & -1 & -1 \\ \hline 1 & 2 & 2 & -2 \\ 1 & 2 & -1 & 2 \\ 2 & 1 & 1 & 1 \end{pmatrix} \begin{matrix} -1 \; -1 \; -2 \\ + \end{matrix} \sim \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 3 & 3 & -1 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & 3 & 3 \end{pmatrix} \begin{matrix} -1 \; -1 \end{matrix} \sim \begin{pmatrix} 1 & -1 & -1 & -1 \\ 0 & 3 & 3 & -1 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

4 LI vectors

$\Longrightarrow$ the vectors v, u_1, u_2, u_3 are linearly independent $\longrightarrow$
the vector v can not be expressed as the linear combination of the vectors u_1, u_2, u_3

$\Longrightarrow$ the exercise has <u>no solution</u>

c) $u_1 = (1,3,5)$, $u_2 = (3,9,15)$, $u_3 = (1,0,2)$; $v = (8,25,40)$

$$\begin{array}{c} v \\ S\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array} \begin{pmatrix} 8 & 25 & 40 \\ \hline 1 & 3 & 5 \\ \cancel{3} & \cancel{9} & \cancel{15} \\ 1 & 0 & 2 \end{pmatrix} \begin{matrix} + \\ -8 \; -1 \\ + \end{matrix} \sim \begin{pmatrix} 8 & 25 & 40 \\ 0 & 1 & 0 \\ 0 & -3 & -3 \end{pmatrix} \begin{matrix} 3 \\ + \end{matrix} \sim \begin{pmatrix} 8 & 25 & 40 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

3 LI vectors

$u_2 = 3 \cdot u_1$,
the vector u_2 can be omitted

$\Longrightarrow$ the vectors u_1, u_3, v are linearly independent $\Longrightarrow$
the vector v can not be expressed as the linear combination of the vectors u_1, u_3

$\Longrightarrow$ the exercise has <u>no solution</u>

5) Exercise: Calculate $\langle u\rangle_M$ or $\langle v\rangle_M$.

a) $u=(-10,7,4)$, $M=\{\overset{u_1}{(2,1,3)}, \overset{u_2}{(-3,1,-2)}, \overset{u_3}{(5,-2,4)}\}$

$u = a u_1 + b u_3 + c u_3$

$(-10,7,-4) = a\,(2,1,3) + b\,(-3,1,-2) + c\,(5,-2,4)$

$(-10,7,-4) = (2a-3b+5c,\ a+b-2c,\ 3a-2b+4c)$

$$\begin{aligned} -10 &= 2a-3b+5c \quad)+ \\ 7 &= a+b-2c \quad -2 \qquad -3 \quad)+ \\ -4 &= 3a-2b+4c \end{aligned}$$

$$\begin{aligned} -24 &= -5b+9c \\ -25 &= -5b+10c \quad - \\ \hline -1 &= c \end{aligned}$$

$-24 = -5b+9c$

$-24-9c = -5b$

$b = \dfrac{24+9c}{5} = \underline{\underline{3}}$

$7 = a+b-2c$

$a = 7-b+2c$

$a = 7-3-2 = \underline{\underline{2}}$

$\langle u\rangle_M = (a,b,c) = (2,3,-1)$

b) $u=(29,12,5)$, $M=\{\overset{u_1}{(-3,2,-4)}, \overset{u_2}{(5,-3,2)}, \overset{u_3}{(0,6,-3)}\}$

$u = a u_1 + b u_2 + c u_3$

$(29,12,5) = a\,(-3,2,-4) + b\,(5,-3,2) + c\,(0,6,-3)$

$(29,12,5) = (-3a+5b,\ 2a-3b+6c,\ -4a+2b-3c)$

$$\begin{aligned} 29 &= -3a+5b \\ 12 &= 2a-3b+6c \quad)+ \\ 5 &= -4a+2b-3c \quad 2 \end{aligned}$$

$$\begin{aligned} 29 &= -3a+5b \\ 22 &= -6a+b \quad -5 \\ \hline 81 &= 27a \\ 3 &= a \end{aligned}$$

$b = 22+6a$

$b = 40$

$5 = -4a+2b-3c$

$c = \dfrac{-5-4a+2b}{3} = \dfrac{-5-12+80}{3} = \dfrac{63}{3} = \underline{\underline{21}}$

$\langle u\rangle_M = (a,b,c) = (3,40,21)$

c) $u=(-5,17,-11),\ M=\{\overset{u_1}{(1,2,1)},\overset{u_2}{(3,-2,7)},\overset{u_3}{(11,-2,23)}\}$

$u = au_1 + bu_2 + cu_3$

$(-5,17,-11) = a(1,2,1) + b(3,-2,7) + c(11,-2,23)$

$(-5,17,-11) = (a+3b+11c,\ 2a-2b-2c,\ a+7b+23c)$

$$\begin{aligned} -5 &= a+3b+11c && \cdot(-2))+ \quad \cdot(-1))+ \\ 17 &= 2a-2b-2c \\ 11 &= a+7b+23c \\ \hline 27 &= -8b-24c &&)+ \\ 16 &= +4b+12c && \cdot 2 \\ \hline 59 &= 0 \end{aligned}$$

it is not OK $\longrightarrow$ there is no solution! We can not find a,b,c.

Where is a problem?

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -2 & 7 \\ 11 & -2 & 23 \end{pmatrix} \begin{matrix} -3)+ \\ -11)+ \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -8 & 4 \\ 0 & -24 & 12 \end{pmatrix} -3)+ \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -8 & 4 \\ \boxed{0 \quad 0 \quad 0} \end{pmatrix} \Longrightarrow$$

u_1, u_2, u_3 are linearly dependent

$\Longrightarrow$ M is no basis, M is only the spanning set, the coordinates are defined only with the respect of the basis

$\Longrightarrow \langle u \rangle_M$ do not exist

try to calculate d, e, f !

g) $u = (-1,0,16,15),\ M = \{\overset{u_1}{(1,2,3,4)},\overset{u_2}{(4,3,2,1)},\overset{u_3}{(3,1,0,2)}\}$

$u = au_1 + bu_2 + cu_3$

$(-1,0,16,15) = a(1,2,3,4) + b(4,3,2,1) + c(3,1,0,2)$

$(-1,0,16,15) = (a+4b+3c,\ 2a+3b+c,\ 3a+2b,\ 4a+b+2c)$

$$\begin{aligned} -1 &= a+4b+3c \\ 0 &= 2a+3b+c && -3)+ \\ 16 &= 3a+2b \\ 15 &= 4a+b+2c \end{aligned}$$

$$\begin{aligned} -1 &= -5a-5b && 3 \\ 16 &= 3a+2b && 5)+ \\ \hline 77 &= -5b \end{aligned}$$

$\underline{b = -\frac{77}{5}}$

$a = \frac{1-5b}{5} = \frac{1}{5} + \frac{77}{5} = \underline{\frac{78}{5}}$

$c = -2a-3b = \underline{\frac{75}{5}}$

examination for the fourth equation

$LS = 15$

$RS = 4a+b+2c = 4\cdot\frac{78}{5} - \frac{77}{5} + \frac{150}{5} = 77 \qquad 77 \neq 15$

$\longrightarrow$ the system of the linear equations has no solution $\Longrightarrow$ the a, b, c do not exist $\Longrightarrow$ we can not find $\langle u \rangle_M$, the exercise has no solution

[The vector u can not be expressed as the linear combination of the vectors u_1, u_2, u_3; the vector $u \notin V$, the vector space V is done by its basis M.]

b) $u = (0,0,1,5)$, $M = \{ \overset{u_1}{(1,2,3,4)}, \overset{u_2}{(4,3,2,1)}, \overset{u_3}{(3,1,0,2)} \}$

$u = a u_1 + b u_2 + c u_3$

$(0,0,1,5) = a(1,2,3,4) + b(4,3,2,1) + c(3,1,0,2)$

$(0,0,1,5) = (a+4b+3c,\ 2a+3b+c,\ 3a+2b,\ 4a+b+2c)$

$0 = a+4b+3c$
$0 = 2a+3b+c \quad -3)+$
$1 = 3a+2b$
$5 = 4a+b+2c$

$0 = -5a-5b \quad 3)+$
$1 = 3a+2b \quad 5$

$5 = -5b$
$\underline{\underline{b=-1}}$

$a = -b = \underline{\underline{1}}$

$c = -2a-3b = -2+3 = \underline{\underline{1}}$

!

LS=5
RS=4a+b+2c= 4−1+2=5
$\underline{\underline{LS=RS=5}}$

$\longrightarrow$ the system of the linear equations has the solution $\Longrightarrow$ the a, b, c were found

$\underline{\underline{\langle u \rangle_M = (a,b,c) = (1,-1,1)}}$

try to calculate i, j, k

c) $u = (7,9,2)$, $v = 3(0,1,2) + 4(1,0,2) + 5(1,2,0)$, $M = \{ \overset{u_1}{(0,1,2)}, \overset{u_2}{(1,0,2)}, \overset{u_3}{(1,2,0)} \}$

$u = a u_1 + b u_2 + c u_3$

$(7,9,2) = a(0,1,2) + b(1,0,2) + c(1,2,0)$

$(7,9,2) = (b+c,\ a+2c,\ 2a+2b)$

$$7 = b + c \quad -2$$
$$9 = a + 2c \quad)+$$
$$2 = 2a + 2b$$

$$-5 = a - 2b \quad)+$$
$$2 = 2a + 2b$$

$$-3 = 3a \qquad b = 1 - a = 1 + 1 = \underline{\underline{2}} \qquad c = 7 - b = 7 - 2 = \underline{\underline{5}}$$
$$\underline{\underline{a = -1}}$$

$$\langle u \rangle_M = (a, b, c) = \underline{\underline{(-1, 2, 5)}}$$

$$v = 3 \cdot (0, 1, 2) + 4 \cdot (1, 0, 2) + 5 \cdot (1, 2, 0) = (9, 13, 14)$$

and $v = a u_1 + b u_2 + c u_3$

$$v = a\,(0, 1, 2) + b\,(1, 0, 2) + c\,(1, 2, 0) = 3 \cdot (0, 1, 2) + 4 \cdot (1, 0, 2) + 5 \cdot (1, 2, 0)$$

- Without our calculation we see the coordinates of the vector v with the respect of the basis M (we use the definition of the coordinates):

$$\underline{\underline{\langle v \rangle_M = (3, 4, 5)}}$$

- the calculation

$$(9, 13, 14) = a\,(0, 1, 2) + b\,(1, 0, 2) + c\,(1, 2, 0)$$
$$(9, 13, 14) = (b + c,\ a + 2c,\ 2a + 2b)$$

$$9 = b + c \quad -2$$
$$13 = a + 2c \quad)+$$
$$14 = 2a + 2b$$

$$-5 = a - 2b$$
$$14 = 2a + 2b$$

$$9 = 3a \qquad b = 7 - a = 7 - 3 = \underline{\underline{4}} \qquad c = 9 - b = 9 - 4 = \underline{\underline{5}}$$
$$\underline{\underline{a = 3}}$$

$$\underline{\underline{\langle v \rangle_M = (3, 4, 5)}}$$

We obtained the same result.