Eigenvalues and eigenvectors

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1) Exercise: Calculate eigenvalues and eigenvectors of the following matrices

a) $A = \begin{pmatrix} 2 & 5 \\ -2 & +3 \end{pmatrix}$

$$\det(A-\lambda E) = \begin{vmatrix} 2-\lambda & 5 \\ -2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)+10 = 6-3\lambda-2\lambda+\lambda^2+10 =$$
$$= \lambda^2 - 5\lambda + 16$$

$$\lambda^2 - 5\lambda + 16 = 0$$
$$\lambda_{1,2} = \frac{5 \pm \sqrt{25-64}}{2} = \frac{5 \pm \sqrt{-39}}{2} = \frac{5 \pm i\sqrt{39}}{2}$$

- In $\mathbb{R}$, A has no eigenvalues.
- In $\mathbb{C}$ $\quad \lambda_1 = \frac{5+i\sqrt{39}}{2} \qquad \lambda_2 = \frac{5-i\sqrt{39}}{2}$

- $\lambda_1 = \frac{5+i\sqrt{39}}{2}$

$$(A-\lambda_1 E)u_1 = o \qquad u_1 \neq o$$

$$\begin{pmatrix} 2-\frac{5}{2}-\frac{i}{2}\sqrt{39} & 5 \\ -2 & 3-\frac{5}{2}-\frac{i}{2}\sqrt{39} \end{pmatrix} \sim \begin{pmatrix} -\frac{1}{2}-\frac{i}{2}\sqrt{39} & 5 \\ -2 & \frac{1}{2}-\frac{i}{2}\sqrt{39} \end{pmatrix} \begin{array}{l} \cdot 2 \\ 2 \end{array} \sim$$

$$\begin{pmatrix} -1-i\sqrt{39} & 10 \\ -4 & 1-i\sqrt{39} \end{pmatrix} \begin{array}{l} 4 \\ -1-i\sqrt{39} \end{array} \sim)+ \begin{pmatrix} -1-i\sqrt{39} & 10 \\ 0 & 0 \end{pmatrix} \qquad u_1 = (10, +1+i\sqrt{39})$$

- $\lambda_2 = \frac{5-i\sqrt{39}}{2}$

$$(A-\lambda E)u_2 = o \qquad u_2 \neq o$$

$$\begin{pmatrix} 2-\frac{5}{2}+\frac{i}{2}\sqrt{39} & 5 \\ -2 & 3-\frac{5}{2}+\frac{i}{2}\sqrt{39} \end{pmatrix} \sim \begin{pmatrix} -\frac{1}{2}+\frac{i}{2}\sqrt{39} & 5 \\ -2 & \frac{1}{2}+\frac{i}{2}\sqrt{39} \end{pmatrix} \begin{array}{l} \cdot 2 \\ 2 \end{array} \sim \begin{pmatrix} -1+i\sqrt{39} & 10 \\ -4 & 1+i\sqrt{39} \end{pmatrix} \begin{array}{l} 4 \\ -1+i\sqrt{39} \end{array} \sim$$

$$\begin{pmatrix} -1+i\sqrt{39} & 10 \\ 0 & 0 \end{pmatrix} \qquad u_2 = (10, 1-i\sqrt{39})$$

b) $\begin{pmatrix} 1 & -3 \\ -4 & 6 \end{pmatrix}$ 2)

$$\det(A-\lambda E) = \begin{vmatrix} 1-\lambda & -3 \\ -4 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) - 12 = 6 - \lambda - 6\lambda + \lambda^2 - 12 =$$
$$= \lambda^2 - 7\lambda - 6$$

$$\lambda^2 - 7\lambda - 6 = 0$$
$$\lambda_{1,2} = \frac{7 \pm \sqrt{49+24}}{2} = \frac{7 \pm \sqrt{73}}{2}$$

$$\underline{\underline{\lambda_1 = \frac{7+\sqrt{73}}{2}}} \qquad \underline{\underline{\lambda_2 = \frac{7-\sqrt{73}}{2}}}$$

$$\lambda_1 = \frac{7+\sqrt{73}}{2}$$

$(A-\lambda_1 E)u_1 = o \quad u_1 \neq o$

$$\begin{pmatrix} 1-\frac{7}{2}-\frac{\sqrt{73}}{2} & -3 \\ -4 & 6-\frac{7}{2}-\frac{\sqrt{73}}{2} \end{pmatrix} \sim \begin{pmatrix} -\frac{5}{2}-\frac{\sqrt{73}}{2} & -3 \\ -4 & \frac{5}{2}-\frac{\sqrt{73}}{2} \end{pmatrix} \begin{matrix} \cdot 2 \\ \cdot 2 \end{matrix} \sim \begin{pmatrix} -5-\sqrt{73} & -6 \\ -8 & +5-\sqrt{73} \end{pmatrix} \begin{matrix} 8 \\ -5-\sqrt{73} \end{matrix}$$

$$\sim \begin{pmatrix} -5-\sqrt{73} & -6 \\ 0 & 0 \end{pmatrix} \qquad \underline{\underline{u_1 = (6, -5-\sqrt{73})}}$$

$$\lambda_2 = \frac{7-\sqrt{73}}{2}$$

$(A-\lambda_2 E)u_2 = o \quad u_2 \neq o$

$$\begin{pmatrix} 1-\frac{7}{2}+\frac{\sqrt{73}}{2} & -3 \\ -4 & 6-\frac{7}{2}+\frac{\sqrt{73}}{2} \end{pmatrix} \sim \begin{pmatrix} -\frac{5}{2}+\frac{\sqrt{73}}{2} & -3 \\ -4 & \frac{5}{2}+\frac{\sqrt{73}}{2} \end{pmatrix} \sim \begin{pmatrix} -5+\sqrt{73} & -6 \\ -8 & 5+\sqrt{73} \end{pmatrix} \begin{matrix} 8 \\ -5+\sqrt{73} \end{matrix}$$

$$\sim \begin{pmatrix} -5+\sqrt{73} & -6 \\ 0 & 0 \end{pmatrix} \qquad \underline{u_2 = (6, -5+\sqrt{73})}$$

c) $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$

$$\det(A-\lambda E) = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 2 = 12 - 7\lambda + \lambda^2 - 2 =$$
$$= \lambda^2 - 7\lambda + 10$$

$\lambda^2 - 7\lambda + 10 = 0$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49-40}}{2} = \frac{7 \pm 3}{2} = \begin{cases} 5 \\ 2 \end{cases}$$

$\underline{\underline{\lambda_1 = 5}}$ $\underline{\underline{\lambda_2 = 2}}$

$\lambda_1 = 5$

$(A - \lambda_1 E)\, u_1 = 0 \quad u_1 \neq 0$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}2 \sim \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \qquad \underline{\underline{u_1 = (1,1)}}$$

$\lambda_2 = 2$

$(A - \lambda_2 E) \cdot u_2 = 0 \quad u_2 \neq 0$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}-1 \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \qquad \underline{\underline{u_2 = (-2,1)}}$$

Try to solve d, e, f, g.

2) Exercise : Calculate eigenvalues and eigenvectors of the following matrices

a)
$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\det(A-\lambda E) = \begin{vmatrix} -\lambda & -1 & 1 \\ 1 & -2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = -\lambda(2+\lambda)(1+\lambda) + 0 + 0 - 0 - 0 - (1+\lambda) =$$
$$(1+\lambda)\left[-2\lambda - \lambda^2 - 1\right] = -(1+\lambda)^3$$

$$\underline{\underline{\lambda_1 = \lambda_2 = \lambda_3 = -1}}$$

$\lambda_1 = -1$

$(A - \lambda_1 E) u_1 = o \quad u_1 \neq o$

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{u_1 = (-1, 0, 1)} \quad \underline{\underline{u_2 = (1, 1, 0)}}$$

b) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix} = A$

$$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2 + 1 + 0 - 0 + 2(2-\lambda) - (3-\lambda) =$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 9) + 1 + 4 - 2\lambda - 3 + \lambda =$$

$$= 2\lambda^2 - \lambda^3 - 12\lambda + 6\lambda^2 + 18 - 9\lambda + 2 - \lambda =$$

$$= \underline{\underline{-\lambda^3 + 8\lambda^2 - 22\lambda + 20}}$$

candidates of the roots $+1, -1, \boxed{+2}, -2, +4, -4, +5, -5, 10, -10, 20, -20$

$\cancel{1}$ $\quad -1 + 8 - 22 + 20 \neq 0$

$\cancel{-1}$ $\quad 1 + 8 + 22 + 20 \neq 0$

$2 \quad -8 + 32 - 44 + 20 = 0$!! $\qquad \lambda_1 = 2$

$$(\lambda^3 - 8\lambda^2 + 22\lambda - 20) : (\lambda - 2) = \lambda^2 - 6\lambda + 10$$
$$\frac{-(\lambda^3 - 2\lambda^2)}{-6\lambda^2 + 22\lambda - 20}$$
$$\frac{-(-6\lambda^2 + 12\lambda)}{10\lambda - 20}$$
$$\frac{-(10\lambda - 20)}{0}$$

$\lambda^2 - 6\lambda + 10 = 0$

$$\lambda_2, \lambda_3 = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm 2i}{2} =$$

$$= 3 \pm i$$

$$\underline{\underline{\lambda_1 = 2}} \qquad \underline{\underline{\lambda_2 = 3 + i}} \qquad \underline{\underline{\lambda_3 = 3 - i}}$$

$\lambda_1 = 2$

$(A - \lambda_1 E) u_1 = o \quad u_1 \neq o$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{u_1 = (1, 0, 1)}}$$

$\lambda_2 = 3+i$

$(A-\lambda_2 E)u_2 = o \quad u_2 \neq o$

$$\begin{pmatrix} -1-i & 1 & 0 \\ 1 & -i & -1 \\ -1 & 2 & -i \end{pmatrix} \begin{matrix} \\ \\ + \end{matrix} \sim \begin{pmatrix} -1-i & 1 & 0 \\ 1 & -i & -1 \\ 0 & 2-i & -1-i \end{pmatrix} \begin{matrix} + \\ 1+i \\ \end{matrix} \begin{pmatrix} 1 & -i & -1 \\ 0 & 2-i & -1-i \\ 0 & 2-i & -1-i \end{pmatrix} \begin{matrix} \\ -1 \\ + \end{matrix} \sim \begin{pmatrix} 1 & -i & -1 \\ 0 & 2-i & -1-i \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{u_2 = (1,\ 1+i,\ 2-i)}}$$

$\lambda_3 = 3-i$

$(A-\lambda_3 E)u_3 = o \quad u_3 \neq o$

$$\begin{pmatrix} -1+i & 1 & 0 \\ 1 & +i & -1 \\ -1 & 2 & i \end{pmatrix} \begin{matrix} 1-i \\ + \\ + \end{matrix} \sim \begin{pmatrix} -1+i & 1 & 0 \\ 0 & 2+i & -1+i \\ 0 & 2+i & -1+i \end{pmatrix} \begin{matrix} \\ -1 \\ + \end{matrix} \sim \begin{pmatrix} -1+i & 1 & 0 \\ 0 & 2+i & -1+i \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{u_3 = \left(\tfrac{1+3i}{2},\ 1-i,\ 2+i\right)}}$$

Try to solve c, d, e, g, h, i, j, k.

Calculate Jordan canonical form of the matrices

3) Exercise: Calculate Jordan canonical form of the matrices

a) $\underline{A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}}$

$$\det(A-\lambda E) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)-4 = -1+\lambda-\lambda+\lambda^2-4 = \lambda^2-5$$

$\lambda^2 - 5 = 0 \qquad \underline{\underline{\lambda_1 = \sqrt{5}}} \qquad \underline{\underline{\lambda_2 = -\sqrt{5}}}$

$$\underline{\underline{J = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{pmatrix}}}$$

b) $\underline{A = \begin{pmatrix} 2 & 5 \\ 5 & 3 \end{pmatrix}}$

$$\det(A-\lambda E) = \begin{vmatrix} 2-\lambda & 5 \\ 5 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)-25 = 6-3\lambda-2\lambda+\lambda^2-25 =$$
$$= \lambda^2 - 5\lambda - 19$$

$$\lambda^2 - 5\lambda - 19 = 0$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{25+76}}{2} = \frac{5 \pm \sqrt{101}}{2}$$

$$\underline{\underline{\lambda_1 = \frac{5+\sqrt{101}}{2}}} \qquad \underline{\underline{\lambda_2 = \frac{5-\sqrt{101}}{2}}}$$

$$\underline{\underline{J = \begin{pmatrix} \frac{5+\sqrt{101}}{2} & 0 \\ 0 & \frac{5-\sqrt{101}}{2} \end{pmatrix}}}$$

c) $\underline{A = \begin{pmatrix} 1 & 0 \\ -4 & 6 \end{pmatrix}}$

$$\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 0 \\ -4 & 6-\lambda \end{vmatrix} = (1-\lambda)(6-\lambda) + 0 = (1-\lambda)(6-\lambda)$$

$$(1-\lambda)(6-\lambda) = 0$$

$$\underline{\underline{\lambda_1 = 1}} \qquad \underline{\underline{\lambda_2 = 6}}$$

$$\underline{\underline{J = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}}}$$

Try to calculate d, e.

4) Exercise: Calculate Jordan canonical form of the matrices

a) $\underline{A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}$

$$\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 + 0 + 0 - (1-\lambda) - (1-\lambda) - 0 =$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 2\right] = (1-\lambda)(\lambda^2 - 2\lambda - 1)$$

$\lambda_1 = 1 \qquad \lambda_{2,3} = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+\sqrt{2} & 0 \\ 0 & 0 & 1-\sqrt{2} \end{pmatrix}$$

b) $A = \begin{pmatrix} 5 & -2 & 1 \\ 5 & -1 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

$$\det(A-\lambda E) = \begin{vmatrix} 5-\lambda & -2 & 1 \\ 5 & -1-\lambda & 1 \\ 2 & -1 & 2-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda)(2-\lambda) - 4 - 5 + 2(1+\lambda) + 5 - \lambda + 10(2-\lambda) =$$

$$= (-5+\lambda-5\lambda+\lambda^2)(2-\lambda) - 9 + 2 + 2\lambda + 5 - \lambda + 20 - 10\lambda =$$

$$= (\lambda^2-4\lambda-5)(2-\lambda) + 18 - 9\lambda = 2\lambda^2 - 8\lambda - 10 - \lambda^3 + 4\lambda^2 + 5\lambda$$

$$+ 18 - 9\lambda = -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = -(\lambda^3 - 6\lambda^2 + 12\lambda - 8)$$

$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$

candidates of the roots: $1, -1, 2, -2, 4, -4, 8, -8$

$\cancel{1} \quad 1 - 6 + 12 - 8 \neq 0$

$-\cancel{1} \quad -1 - 6 - 12 - 8 \neq 0$

$2 \quad 8 - 24 + 24 - 8 = 0$! $\qquad \lambda_1 = 2$

$(\lambda^3 - 6\lambda^2 + 12\lambda - 8) : (\lambda - 2) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 \qquad \lambda_2 = \lambda_3 = 2$

$-(\lambda^3 - 2\lambda^2)$

$\quad -4\lambda^2 + 12\lambda - 8$

$\quad -(-4\lambda^2 + 8\lambda)$

$\qquad 4\lambda - 8$

$\qquad -(4\lambda - 8)$

$\qquad\quad 0$

$$J = \begin{pmatrix} 2 & 0 & 0 \\ \sqcup & 2 & 0 \\ 0 & \sqcup & 2 \end{pmatrix}$$

we must calculate the number of the linearly independent eigenvectors associated with eigenvalues $\lambda_1 = \lambda_2 = \lambda_3$

8)

$\lambda_1 = \lambda_2 = \lambda_3 = 2$

$(A - \lambda_1 E) u_1 = 0 \quad u_1 \neq 0$

$$\begin{pmatrix} 3 & -2 & 1 \\ 5 & -3 & 1 \\ 2 & -1 & 0 \end{pmatrix} \begin{matrix} -5 \\ 3 \\ \\ \end{matrix} \sim \begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix}$$

$\text{rank}(A - \lambda_1 E) = 2$
$\rightarrow \dim V_2 = 3 - 2 = 1$
$\rightarrow$ only one eigenvector
$\rightarrow$ only " one Jordan block

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

c) $A = \begin{pmatrix} -2 & -1 & 1 \\ 5 & -1 & 4 \\ 5 & 1 & 2 \end{pmatrix}$

$$\det(A - \lambda E) = \begin{vmatrix} -2-\lambda & -1 & 1 \\ 5 & -1-\lambda & 4 \\ 5 & 1 & 2-\lambda \end{vmatrix} = +(2+\lambda)(1+\lambda)(2-\lambda) + 5 - 20 + 5(1+\lambda) + 4(2+\lambda) + 5(2-\lambda) =$$

$$= (4-\lambda^2)(1+\lambda) - 15 + 5 + 5\lambda + 8 + 4\lambda + 10 - 5\lambda =$$

$$= 4 - \lambda^2 + 4\lambda - \lambda^3 + 8 + 4\lambda = -\lambda^3 - \lambda^2 + 8\lambda + 12$$

$\lambda^3 + \lambda^2 - 8\lambda - 12 = 0$

candidates of the roots: $1, -1, 2, -2, 3, -3, 4, -4, 12, -12$

$\cancel{1} \quad 1 + 1 - 8 - 12 \neq 0$

$-\cancel{1} \quad -1 + 1 + 8 - 12 \neq 0$

$\cancel{2} \quad 8 + 4 - 32 - 12 \neq 0$

$-\cancel{2} \quad -8 + 4 + 32 - 12 \neq 0$

$3 \quad 27 + 9 - 24 - 12 = 0$! $\quad \lambda_1 = 3$

$(\lambda^3 + \lambda^2 - 8\lambda - 12) : (\lambda - 3) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 \qquad \lambda_2 = \lambda_3 = -2$

$-(\lambda^3 - 3\lambda^2)$

$4\lambda^2 - 8\lambda - 12$

$-(4\lambda^2 - 12\lambda)$

$4\lambda - 12$

$-(4\lambda - 12)$

0

$$J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 4 & -2 \end{pmatrix}$$

we must calculate the number of the linearly independent eigenvectors associated with eigenvalues $\lambda_2 = \lambda_3 = -2$

$\lambda_2 = \lambda_3 = -2$

$(A - \lambda_2 E) u_2 = o \quad u_2 \neq o$

$$\begin{pmatrix} 0 & -1 & 1 \\ 5 & 1 & 4 \\ 5 & 1 & 4 \end{pmatrix} \sim \begin{pmatrix} 5 & 1 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

rank $(A - \lambda_2 E) = 2$

$\dim V_{-2} = 3 - 2 = 1$

$\rightarrow$ only „one" eigenvector

$\rightarrow$ for the $\lambda_2 = \lambda_3 = -2$, there is only one block

$$J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

Try to calculate d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v

5) Exercise: Calculate Jordan canonical form of the matrix A and determine, if possible, a matrix P satisfying $J = P^{-1} A P$.

a)
$$A = \begin{pmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 18 & -9 & -3 \end{pmatrix}$$

$$\det |A - \lambda E| = \begin{vmatrix} 12-\lambda & -6 & -2 \\ 18 & -9-\lambda & -3 \\ 18 & -9 & -3-\lambda \end{vmatrix} = (12-\lambda)(9+\lambda)(3+\lambda) + 324 + 324 + 36(-9-\lambda) - 27(12-\lambda) + 108(-3-\lambda)$$

$$= (108 - 9\lambda + 12\lambda - \lambda^2)(3+\lambda) + 648 - 324 - 36\lambda - 324 + 27\lambda - 324 - 108\lambda = 324 - 27\lambda + 36\lambda - 3\lambda^2 + 108\lambda - 9\lambda^2 + 12\lambda^2 - \lambda^3$$

$$- 117\lambda - 324 = -\lambda^3$$

$\lambda_1 = \lambda_2 = \lambda_3 = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 0$

$(A - \lambda_1 E) u_1 = 0 \quad u_1 \neq 0$

$$\begin{pmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 18 & -9 & -3 \end{pmatrix} \begin{matrix} :2 \\ :3 \\ :3 \end{matrix} \sim \begin{pmatrix} 6 & -3 & -1 \\ 6 & -3 & -1 \\ 6 & -3 & -1 \end{pmatrix} \begin{matrix} \\ -1 \\ -1 \end{matrix} \sim \begin{pmatrix} 6 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$u_1 = (\frac{1}{2}, 1, 0) \sim (1, 2, 0)$

$u_2 = (\frac{1}{6}, 0, 1) \sim (1, 0, 6)$

$$J = \begin{pmatrix} \boxed{0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Looking for the third vector

$? \ (A - \lambda_1 E) u_3 = u_1$

$$\left(\begin{array}{ccc|c} 12 & -6 & -2 & 1 \\ 18 & -9 & -3 & 2 \\ 18 & -9 & -3 & 0 \end{array}\right) \begin{matrix} -3 & -3 \\ 2 & 1 \\ & 2 \end{matrix} \sim \left(\begin{array}{ccc|c} 12 & -6 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right)$$

the solution does not exist

$? \ (A - \lambda_1 E) u_3 = u_2$

$$\left(\begin{array}{ccc|c} 12 & -6 & -2 & 1 \\ 18 & -9 & -3 & 0 \\ 18 & -9 & -3 & 6 \end{array}\right) \begin{matrix} -3 & -3 \\ 2 & \\ & 2 \end{matrix} \sim \left(\begin{array}{ccc|c} 12 & -6 & -2 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 9 \end{array}\right)$$

the solution does not exist

$?? \ (A - \lambda_1 E) u_3 = u$

where „u" is the right linear combination of vectors u_1 and u_2

↳ we see, that the „right" combination is $3 \cdot u_1 + 1 u_2 = (4, 6, 6)$

$$\left(\begin{array}{ccc|c} 12 & -6 & -2 & 4 \\ 18 & -9 & -3 & 6 \\ 18 & -9 & -3 & 6 \end{array}\right) \begin{matrix} :2 \\ :3 \\ :3 \end{matrix} \sim \left(\begin{array}{ccc|c} 6 & -3 & -1 & 2 \\ 6 & -3 & -1 & 2 \\ 6 & -3 & -1 & 2 \end{array}\right) \begin{matrix} \\ -1 \\ -1 \end{matrix} \sim \left(\begin{array}{ccc|c} 6 & -3 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underline{u = (\tfrac{1}{3}, 0, 0)}$$

$$J = \begin{pmatrix} \boxed{0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad P = \begin{array}{c} \begin{pmatrix} 1 & \frac{1}{3} & 4 \\ 2 & 0 & 6 \\ 0 & 0 & 6 \end{pmatrix} \\ u_1 \quad u_3 \leftarrow u \end{array}$$

Try to calculate b, c, d, e, f, g, i, j, k, l