

Quadratic forms

1) Exercise: Decide whether f is a quadratic form. If yes, write its matrix, calculate its signature

a) $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$

$$f(x_1, x_2, x_3) = \sum_{i,j=1}^3 a_{ij}x_i x_j \quad \text{OK}$$

matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

signature

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{matrix} -2 & -1 \\ + \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{matrix} -2 \\ + \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{matrix} -3 \\ + \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & -4 \end{pmatrix} \begin{matrix} -3 \\ -3 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$$\text{sgn } A = (2, 1, 0)$$

b) $f(x_1, x_2, x_3) = x_1^2 - 2x_2^2 + x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3$

$$f(x_1, x_2, x_3) = \sum_{i,j=1}^3 a_{ij}x_i x_j$$

matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

signature

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{matrix} -1 \\ + \\ -2 \end{matrix} \sim \begin{pmatrix} 1 & 1 & 2 \\ 0 & -3 & -1 \\ 0 & -1 & -3 \end{pmatrix} \begin{matrix} -1 \\ -2 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & -1 & -3 \end{pmatrix} \begin{matrix} -3 \\ -3 \end{matrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -24 \end{pmatrix} \begin{matrix} -3 \\ + \end{matrix}$$

$$\text{sgn } A = (1, 2, 0)$$

Try to calculate c, d, e, f, g, h, i, j .

2) Exercise: Decide whether the matrices A and B represent quadratic forms. If yes, write their analytic expressions and express their signatures.

$$a) \quad A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

A yes, it is the square and symmetric matrix

$$\begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \xrightarrow{\substack{+ \\ -2 \\ -4}} \begin{pmatrix} 4 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & 2 \end{pmatrix} \xrightarrow{+} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 32 \end{pmatrix} \xrightarrow{+}$$

$$\sim \begin{pmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 32 \end{pmatrix} \quad \text{sgn } A = (2, 1, 0)$$

B - no, it is the square matrix, but it is not a square matrix

$$b) \quad A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 12 & 1 \\ -1 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

A - yes, the symmetric square matrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 12 & 1 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow{+} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 12 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 12 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\substack{+ \\ -12}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 12 & 1 \\ 0 & 0 & -11 \end{pmatrix} \xrightarrow{+} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 132 \end{pmatrix}$$

$$\text{sgn } A = (3, 0, 0)$$

B - no, it is not a square matrix

Try to solve c, d, e, f, g, h, i, j

3) Exercise: Calculate signature of the quadratic forms f and g

3)

a) $f(x_1, x_2, x_3, x_4) = -4x_1x_4$

$g(x_1, x_2) = x_1^2 - 2x_1x_2 + 4x_2^2$

F: $\begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -4 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -4 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ $\text{sgn } f = (1, 1, 2)$

G: $\begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ $\text{sgn } g = (2, 0, 0)$

b) $f(x_1, x_2) = x_1^2 + 26x_2^2 + 10x_1x_2$

$g(x_1, x_2) = x_1^2 + 56x_2^2 + 16x_1x_2$

F: $\begin{pmatrix} 1 & 5 \\ 5 & 26 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\text{sgn } f = (2, 0, 0)$

G: $\begin{pmatrix} 1 & 8 \\ 8 & 56 \end{pmatrix} \sim \begin{pmatrix} 1 & 8 \\ 0 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -8 \end{pmatrix}$ $\text{sgn } g = (1, 1, 0)$

Try to calculate $c_i, d_i, e_i, f_i, g_i, h_i, i, j$