IV. DETERMINANTS

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Definition

If A is an $n \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix},$$

then the determinant of A, denoted $\det A$, is defined by

$$\det A = \sum_{\pi \in S_n} \operatorname{sgn} \pi \cdot a_{1\pi(1)} \cdot a_{2\pi(2)} \cdot \ldots \cdot a_{n\pi(n)}.$$

Notes for better understanding

1. Here π is the symbol for a permutation of the indices of matrix columns. A permutation of $(1, 2, \dots, n)$ is an n-tuple $(m_1, m_2, \dots, m_n)$ that contains each of the numbers $1, \dots, n$ exactly once. The set of all permutations of $(1, 2, \dots, n)$ is denoted S_n. For example, if $S = \{1, 2, 3\}$, then the set S_3 consists of six permutations ($S_3 = \{(1, 2, 3), (1, 3, 2,), (2, 1, 3,), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$). We know that S_n has $n!$ elements ($n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$).
2. The symbol $\operatorname{sgn} \pi$ is called the *sign of permutation* $(m_1, m_2, \dots, m_n)$. The sign of permutation π is defined to be 1 if there is an even number of pairs of integers (j, k) with $1 \le j < k \le n$ such that $m_j > m_k$ and -1 if there is an odd number of such pairs. In other words, the sign of a permutation equals -1 if the natural order has been reversed odd number times. For example the permutations $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$ have their signs 1, -1, -1, 1, -1 and 1.
3. We will denote the determinant of the matrix A by the symbol

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}, \quad \text{or} \quad \det A, \quad \text{or} \quad |A|.$$

4. In some texts, the determinant is defined as a function on the square matrices that is linear as a function of each row and that it changes the sign when two rows are interchanged, so that the determinant is a multilinear and alternating function on the square matrices. To prove that such a function exists and that it is unique is a non-trivial task.
5. Sometimes we will speak about the „rows or columns of a determinant“ instead of a more precise expression „rows or columns of a matrix from which we calculate the determinant“.

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How to calculate determinant of square matrices of „small“ row ranks

1. If A is 1×1 matrix, $A = (a_{11})$, then $\det A = a_{11}$.

2. If A is 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

Example

We have

$$\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} = 2 \cdot 5 - (-1) \cdot 3 = 10 + 3 = 13.$$

3. If A is a 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

then

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

The above described „algorithm“ is called *Sarrus' rule.*

Example

We have

$$\begin{vmatrix} 2 & 3 & 0 \\ -1 & 5 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 2 \cdot 5 \cdot 1 + 3 \cdot 1 \cdot 0 + (-1) \cdot 2 \cdot 0 - 0 \cdot 5 \cdot 0 - 2 \cdot 2 \cdot 1 - (-1) \cdot 3 \cdot 1 =$$

$$= 10 + 0 + 0 - 0 - 4 + 3 = 9.$$

For practical caluculations, the definition of the determinants is not, in general, suitable because $n!$ grows large very rapidly as n increases. For example,

$$\begin{array}{lcccccc} n=2 & & S_2 & = & 2! & = & 2 \\ n=3 & & S_3 & = & 3! & = & 6 \\ n=4 & & S_4 & = & 4! & = & 24 \\ n=5 & & S_5 & = & 5! & = & 120 \end{array}$$

We wish to find a more effective method for our computation.

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General theorems

1. If A is a square matrix with one zero row, then $\det A = 0$.
2. If A is a square matrix with two equal rows, then $\det A = 0$.
3. Suppose that A is a square matrix. If B is the matrix obtained from A by interchanging two rows, then $\det B = -\det A$.
4. For every a square matrix A, $\det A = \det A^t$.
5. Suppose that A is a square matrix. If B is the matrix obtained from A by multiplying a row by a scalar λ, then $\det B = \lambda \cdot \det A$.
6. Suppose that A is a square matrix. If B is the matrix obtained from A by multiplying it by a scalar λ, then $\det B = \lambda^n \cdot \det A$.
7. Suppose that A is a square matrix. If B is the matrix obtained from A by adding, say, λ-times the i-th row to the j-th row then, $\det B = \det A$.
8. The determinant of every upper-triangualar (or lower-triangular) matrix is equal to the product of all diagonal entries, thus $\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdot \ldots \cdot a_{nn}$.
9. The determinant of every diagonal matrix is equal to the product of the diagonal entries, so that $\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdot \ldots \cdot a_{nn}$.

Try to prove these theorems using only the definition of the determinant of a matrix.

Theorem

If A and B are square matrices of the same size, then

$$\det AB = \det A \cdot \det B.$$

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The following result shows how we can calculate determinant of $n \times n$ matrix from determinants of $(n-1) \times (n-1)$ matrices.

Definition

Let A be a matrix of size $n \times n$. By the symbol A_{ij} we mean the matrix of size $(n-1) \times (n-1)$ obtained from the matrix A by omitting its i-th row and j-th column. The determinant of matrix A_{ij} is called the *subdeterminant* of matrix A which corresponds to the entry a_{ij}.

Theorem – Laplace expansion along the i-th row (along the j-th column)

Let A be a matrix of size $n \times n$. Then

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},$$

which is called the *Laplace expansion along the i-th row.* Alternatively

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij},$$

which is called the *Laplace expansion along the j-th column.*

Note the independence on the row (column) chosen. Note also that on the left hand side, there is just determinant of matrix A of size $n \times n$, on the right hand side there is n determinants of matrices of size $(n-1) \times (n-1)$ which come from the matrix A.

Example

Calculate the determinant of a matrix A, where

$$A = \begin{vmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{vmatrix}.$$

The matrix A is of size 4×4, so that for its calculation it is impossible to use Sarrus' algorithm. We use preferably Laplace expansion along the 1-th column because it contains many zero entries.

$$\begin{vmatrix} 1 & 1 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{vmatrix} =$$

$$= (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} + (-1)^{2+1} \cdot 0 \cdot \begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} + (-1)^{3+1} \cdot 0 \cdot \begin{vmatrix} 1 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{vmatrix} + (-1)^{4+1} \cdot 0 \cdot \begin{vmatrix} 1 & 3 & 4 \\ 1 & 1 & 1 \\ 1 & 4 & 0 \end{vmatrix} =$$

$$= 1 \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} + 0 + 0 + 0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 12 + 0 + 0 - 4 - 0 - 3 = 5.$$

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Applications of determinants

In what follows, we will offer the most important applications of determinants.

• The algorithm for calculation of an inverse matrix

Definition

Let A be a matrix of size $n \times n$. We define the *adjugate* matrix of A to be the $n \times n$ matrix $\operatorname{adj} A$ given by

$$(\operatorname{adj} A)_{ij} = (-1)^{i+j} \det A_{ji}.$$

It is very important to note the interchange of the indices in the above definition. The adjugate matrix has the following useful property.

Theorem

Let A be matrix of size $n \times n$, $\operatorname{adj} A$ its adjugate matrix, then

$$A \cdot \operatorname{adj} A = (\det A) \cdot E_{n\times n}.$$

The matrix $(\det A) \cdot E_{n\times n}$ is a diagonal matrix in which every entry on the diagonal is the scalar $(\det A)$. We can use the above result to obtain a convenient way of determinig whether or not a given matrix is invertible, and a new way of computing inverses.

Theorem

A square matrix A is invertible, if and only if $\det A \neq 0$, in which case the inverse is given by

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A.$$

Note

The above result provides a new way how to compute inverse matices.

In particular, one should note the factor $(-1)^{i+j} = \pm 1$. The sign is given according to the scheme

$$\begin{array}{ccccc}
+ & - & + & - & \dots \\
- & + & - & + & \dots \\
+ & - & + & - & \dots \\
- & + & - & + & \dots \\
\dots & \dots & \dots & \dots & \dots
\end{array}$$

Example

Let be A

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 5 & 7 \\ 3 & 2 & 1 \end{pmatrix}.$$

Calculate A^{-1} (if exists).

$$\det A = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 5 & 7 \\ 3 & 2 & 1 \end{vmatrix} = 5 + 0 - 21 - 0 - 0 - 14 = -30 \neq 0.$$

Now we know that the matrix A^{-1} exists. We will calculate the *adjugate* matrix of A.

$$\det A_{11} = \begin{vmatrix} 5 & 7 \\ 2 & 1 \end{vmatrix} = -9 \quad \det A_{21} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1 \quad \det A_{31} = \begin{vmatrix} -1 & 0 \\ 5 & 7 \end{vmatrix} = -7$$

$$\det A_{12} = \begin{vmatrix} 0 & 7 \\ 3 & 1 \end{vmatrix} = -21 \quad \det A_{22} = \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1 \quad \det A_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 7 \end{vmatrix} = 7$$

$$\det A_{13} = \begin{vmatrix} 0 & 5 \\ 3 & 2 \end{vmatrix} = -15 \quad \det A_{23} = \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 5 \quad \det A_{33} = \begin{vmatrix} 1 & -1 \\ 0 & 5 \end{vmatrix} = 5$$

So $\operatorname{adj} A$ is

$$\operatorname{adj} A = \begin{pmatrix} -9 & 1 & -7 \\ 21 & 1 & -7 \\ -15 & -5 & 5 \end{pmatrix}.$$

The inverse matrix of A is

$$A^{-1} = -\frac{1}{30} \cdot \begin{pmatrix} -9 & 1 & -7 \\ 21 & 1 & -7 \\ -15 & -5 & 5 \end{pmatrix}.$$

In order to calculate the inverse matrix of matrix A of size 3 we had to compute one determinant of a matrix A and 9 determinants of adjugate matrices of matrix A.

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- **The algorithm for solving system of linear equations with a regular matrix**

  Theorem (so called Cramer's rule)

Let be $Ax^t = b^t$ non-homogeneous system of linear equations where A is a matrix $n \times n$ of rank n (so that A is a regular matrix), $b \neq o$. Then the system has only one solution $(x_1, x_2, \ldots, x_n)$ given by

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, \ldots, n,$$

where A_i is a matrix which formed from the matrix A by exchanging the i-th column of matrix A by the column b^t.

Example

Solve the following system of linear equations

$$\begin{aligned} x + y + z &= 1, \\ x - y &= 2, \\ x - z &= 0. \end{aligned}$$

We will use Cramer's rule.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 3,$$

$$\det A_x = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 3, \quad \det A_y = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -3, \quad \det A_z = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 0 \end{vmatrix} = 3.$$

$$x = \frac{\det A_x}{\det A} = \frac{3}{3} = 1, \quad y = \frac{\det A_y}{\det A}, = \frac{-3}{3} = -1, \quad z = \frac{\det A_z}{\det A}, = \frac{3}{3} = 1.$$

So the solution of our system of linear equations is $[1, -1, 1]$.

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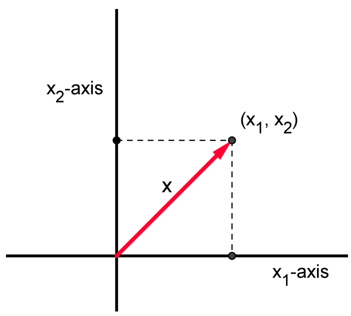
Some geometrical applications of vector spaces, matrices and determinants

• **Dot product**

To motivate the concept of inner product, let us consider vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as arrows with initial point at the origin. The length of a vector $x \in \mathbb{R}^2$ or $\mathbb{R}^3$ is called the *norm* of x, denoted $||x||$. Thus for $x = (x_1, x_2) \in \mathbb{R}^2$,

$$||x|| = \sqrt{x_1^2 + x_2^2}.$$



Similarly, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The generalization to $\mathbb{R}^n$ is obvious.

Definition

We define the *norm* of a vector x, $x = (x_1, x_2, \dots,\ x_n) \in \mathbb{R}^n$, by

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Definition

For all $x, y \in \mathbb{R}^n$ we define the *dot product* of x and y, denoted $x \cdot y$, by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

Notes

1. Note that the *dot product* of two vectors in $\mathbb{R}^n$ is a number, not a vector.
2. Obviously $x \cdot x = ||x||^2$ for all $x \in \mathbb{R}^n$.
3. In particular, $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$, with equality if and only if $x = 0$.
4. Also, if $y \in \mathbb{R}^n$ is fixed, then clearly the map from $\mathbb{R}^n$ to $\mathbb{R}$ sending $x \in \mathbb{R}^n$ to $x \cdot y$ is linear.
5. Furthermore, $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}^n$.

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• Inner product

The notion of inner product is a generalization of the dot product.

Definition

An *inner product* on a vector space V is a function that maps each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in F$ and has the following properties:

1. *positivity*: $\langle v, v\rangle \geq 0$ for all $v \in V$;
2. *definiteness*: $\langle v, v\rangle = 0$ if and only if $v = 0$;
3. *additivity in the first variable*: $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$;
4. *homogenity in the first variable*: $\langle av, w\rangle = a\,\langle v, w\rangle$ for all $a \in F$ and all $v, w \in V$;
5. *conjugate transpose*: $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in V$.

Recall that every real number equals its complex conjugate. Thus if we are dealing with a real vector space, then in the last condition we can simply state that $\langle v, w\rangle = \langle w, v\rangle$ for all $v, w \in V$.

Note

Note in for $\mathbb{R}^n$, the dot product is an inner product.

Definition

An *inner product space* is a vector space V equipped with an inner product on V.

Definition

Two vectors $u, v \in V$ are said to be *orthogonal*, if $\langle u, v\rangle = 0$.

Note that the order of the vectors does not matter because $\langle u, v\rangle = 0$, if and only if $\langle v, u\rangle = 0$. Instead of saying that u and v are orthogonal, sometimes we say that u is orthogonal to v. Clearly o is orthogonal to every vector. Futhermore, o is the only vector that is orthogonal to itself.

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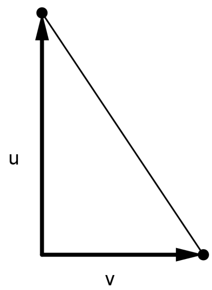
• **Some known theorems**

Pythagorean theorem

If u, v are orthogonal vectors in V, then

$$||u + v||^2 = ||u||^2 + ||v||^2.$$



Cauchy-Schwarz inequality

If u, v are vectors in V, then

$$|\langle u, v\rangle| \leq ||u|| \cdot ||v||.$$

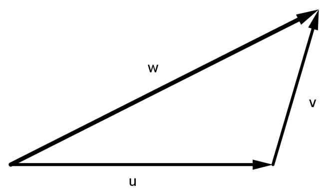
The equality holds if and only if one of u, v is a scalar multiple of the other.

Triangle inequality

If u, v are vectors in V, then

$$||u+v|| \leq ||u|| + ||v||.$$

The equality holds if and only, if one of u, v is a nonnegative multiple of the other.

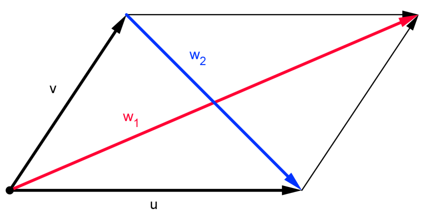


$$\overrightarrow{w} = \overrightarrow{u} + \overrightarrow{v}.$$

Parallelogram equality

If u, v are vectors in V, then

$$||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2).$$



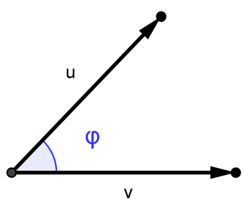
$$\overrightarrow{w_1} = \overrightarrow{u} + \overrightarrow{v}, \qquad \overrightarrow{w_2} = \overrightarrow{u} - \overrightarrow{v}.$$

Definition

If u, v are vectors in V, $u \neq o$ and $v \neq o$, then we define the angle φ between vectors u and v as

$$\cos\varphi = \frac{\langle u, v\rangle}{||u|| \cdot ||v||},$$

where $\varphi \in \langle 0, \pi\rangle$.



Definition

A set of vectors is called *orthonormal* if the vectors are pairwise orthogonal and each vector has norm 1. An *orthonormal basis* of V is an orthonormal set of vectors in V that is also a basis of V.

Example

A standard orthonormal basis of the vector space $\mathbb{R}^n$ is

$$B = \{(1,0,0,\ldots,0),(0,1,0,\ldots,0),(0,0,1,\ldots,0),\ldots,(0,0,0,\ldots,1)\}$$

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Gram-Schmidt theorem

If $\{v_1, v_2, \ldots, v_m\}$ is a linearly independent set of vectors in V, then there exists an orthonormal set $\{e_1, \ldots, e_m\}$ of vectors in V such that

$$\text{span}\,\{v_1, v_2, \ldots, v_j\} = \text{span}\{e_1, \ldots, e_j\}$$

for $j = 1, \ldots, m$.

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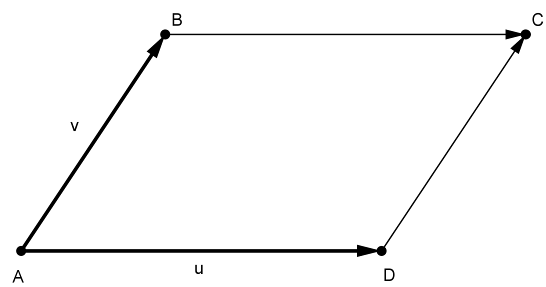
- **Vector product in** $\mathbb{R}^3$

Definition

Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. By the *vector product of* u *and* v we mean the vector $w \in \mathbb{R}^3$ given by

$$u \times v = (u_1, u_2, u_3) \times (v_1, v_2, v_3) = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Note that the vector $u \times v$ is orthogonal to both vectors u and v.



Note that the norm of the vector $u \times v$, that is $||u \times v||$, is equal to the area of parallelogram $ABCD$ given by vectors u and v.

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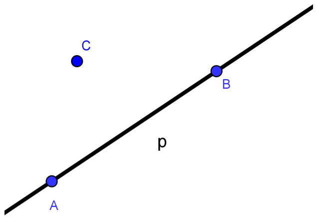
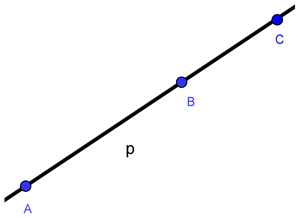
• Some applications of determinants in analytic geometry

Plane

1. Let $A = [x_1, y_1]$, $B = [x_2, y_2]$ and $C = [x_3, y_3]$ be three points in the plane V. Decide whether A, B and C lie on a single straight-line.





Calculate the determinant

$$\det A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

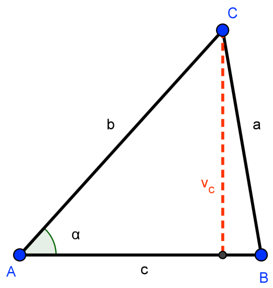
If $\det A = 0$, then the points A, B and C lie on a single straight-line. If $\det A \neq 0$, then the points A, B and C do not lie on any straight-line.

2. Let $A = [x_1, y_1]$ and $B = [x_2, y_2]$ be two points in the plane V. Write an analytic equation (so called general equation) of the straight-line AB.

Calculate the following determinant to obtain the analytic equation of the straight-line AB

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = (y_1 - y_2)x + (x_1 - x_2)y + x_1y_2 - x_2y_1 = 0.$$

3. Let $A = [x_1, y_1]$, $B = [x_2, y_2]$ and $C = [x_3, y_3]$ be three points in the plane V. Calculate the area of the triangle ABC.



Calculate the following determinant to obtain the area of the triangle ABC:

$$S = \frac{1}{2}\left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|.$$

Note

From the secondary school we know that the area of the triangle ABC can be calculated as

$$S = \frac{1}{2} c \cdot v_c = \frac{1}{2} b \cdot c \cdot \sin\alpha = \frac{1}{2} ||(|\overrightarrow{AB}| \times |\overrightarrow{AC}|)||.$$

Space

1. Let $A = [x_1, y_1, z_1]$, $B = [x_2, y_2, z_2]$, $C = [x_3, y_3, z_3]$, $D = [x_4, y_4, z_4]$ be four points in $\mathbb{R}^3$. Decide whether A, B, C and D lie on a single plane.

Calculate the determinant

$$\det A = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

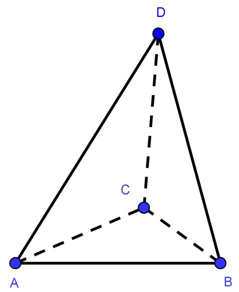
If $\det A = 0$, then the points A, B, C and D lie on a single plane. If $\det A \neq 0$, then the points A, B, C and D do not lie on any plane.

2. Let $A = [x_1, y_1, z_1]$, $B = [x_2, y_2, z_2]$ and $C = [x_3, y_3, z_3]$ be three points in $\mathbb{R}^3$ (not belonging to a single straight-line). Write the analytic equation (so called general equation) of the plane ABC.

Calculate the following determinant to obtain the analytic equation of the plane ABC

$$\det A = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x & y & z & 1 \end{vmatrix} = 0.$$

3. Let $A = [x_1, y_1, z_1]$, $B = [x_2, y_2, z_2]$, $C = [x_3, y_3, z_3]$ and $D = [x_4, y_4, z_4]$ be four points in $\mathbb{R}^3$. Calculate the volume of the tetrahedron $ABCD$.



Calculate the following determinant to obtain the volume of the tetrahedron $ABCD$

$$V = \frac{1}{6} \left| \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \right|.$$

Note

From the secondary school we know that the volume of the tetrahedron $ABCD$ can be calculated as

$$V = \frac{1}{3} S_{ABC} \cdot v = \frac{1}{6} |(|\overrightarrow{AB}| \times |\overrightarrow{AC}|) \cdot |\overrightarrow{AD}|)|,$$

where v is the distance of the point D from the plane ABC and S_{ABC} is the area of the triangle ABC.

Exercises

1. Calculate the following determinants

a)

$$\begin{vmatrix} 2 & 5 \\ -2 & 3 \end{vmatrix}.$$

b)

$$\begin{vmatrix} 1 & -3 \\ -4 & 6 \end{vmatrix}.$$

c)

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}.$$

d)

$$\begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix}.$$

e)

$$\begin{vmatrix} 1 & 0 \\ a & -2 \end{vmatrix}.$$

f)

$$\begin{vmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{vmatrix}.$$

g)

$$\begin{vmatrix} \sin x & -\sin y \\ \cos x & \cos y \end{vmatrix}.$$

h)

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & \sin x \end{vmatrix}.$$

i)

$$\begin{vmatrix} \tan x & -1 \\ 1 & \tan x \end{vmatrix}.$$

j)

$$\begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix}.$$

2. Calculate the following determinants

a)
$$\begin{vmatrix} 3 & -2 & 1 \\ -5 & 3 & 4 \\ 2 & 1 & 3 \end{vmatrix}.$$

b)
$$\begin{vmatrix} 4 & 10 & 1 \\ 0 & 2 & 0 \\ 1 & -3 & 7 \end{vmatrix}.$$

c)
$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{vmatrix}.$$

d)
$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}.$$

e)
$$\begin{vmatrix} 4 & 2 & 1 \\ 3 & -2 & -2 \\ 1 & 0 & 5 \end{vmatrix}.$$

f)
$$\begin{vmatrix} 5 & 0 & -1 \\ 2 & 4 & 0 \\ -3 & 6 & 1 \end{vmatrix}.$$

g)
$$\begin{vmatrix} 1 & 5 & -2 \\ 0 & 2 & -1 \\ -3 & 1 & 1 \end{vmatrix}.$$

h)
$$\begin{vmatrix} -1 & 1 & 1 \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{vmatrix}.$$

i)
$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{vmatrix}.$$

j)
$$\begin{vmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & -1 \end{vmatrix}.$$

k)

$$\begin{vmatrix} 1 & 0 & f \\ u & 1 & k \\ 0 & 1 & k \end{vmatrix}.$$

l)

$$\begin{vmatrix} 0 & a & a \\ a & 0 & a \\ a & a & 0 \end{vmatrix}.$$

m)

$$\begin{vmatrix} a & a & a \\ -a & 0 & a \\ -a & -a & 0 \end{vmatrix}.$$

n)

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}.$$

o)

$$\begin{vmatrix} \sin x & \cos x & 1 \\ \sin y & \cos y & 1 \\ \sin z & \cos z & 1 \end{vmatrix}.$$

3. Calculate the following determinants

a)

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -5 & 2 & -1 & 1 \\ -6 & 5 & 2 & 1 \\ 3 & -1 & 1 & 0 \end{vmatrix}.$$

b)

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix}.$$

c)

$$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}.$$

d)

$$\begin{vmatrix} -1 & -2 & 3 & -1 \\ 2 & 4 & -3 & 2 \\ 1 & 2 & -2 & -1 \\ -2 & 1 & 1 & -2 \end{vmatrix}.$$

e)

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

f)

$$\begin{vmatrix} 2 & 1 & 10 & 2 \\ 2 & 2 & -3 & 2 \\ -1 & 2 & 11 & 1 \\ 1 & 2 & 8 & 1 \end{vmatrix}.$$

g)

$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 4 & 10 \\ 1 & 0 & 3 & -5 \\ 2 & 5 & 2 & 2 \end{vmatrix}.$$

h)

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & 2 & 2 & -2 \end{vmatrix}.$$

i)

$$\begin{vmatrix} -5 & 1 & -4 & 1 \\ 1 & 4 & -1 & 5 \\ -4 & 1 & -8 & -1 \\ 3 & 2 & 6 & 2 \end{vmatrix}.$$

j)

$$\begin{vmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix}.$$

k)

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix}.$$

l)

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix}.$$

m)

$$\begin{vmatrix} 0 & c & 1 & 0 \\ 1 & 0 & a & 0 \\ 0 & b & 0 & 0 \\ 1 & 0 & -c & 1 \end{vmatrix}.$$

n)

$$\begin{vmatrix} 1 & 1 & 1 & a \\ 2 & 1 & 2 & b \\ 1 & -1 & 1 & c \\ 2 & 1 & -2 & d \end{vmatrix}.$$

4. Solve the equations

a)

$$\begin{vmatrix} x^2 & 3 & 2 \\ x & -1 & 1 \\ 0 & 1 & 4 \end{vmatrix} = 0.$$

b)

$$\begin{vmatrix} x^2 & 4 & 9 \\ x & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

c)

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & -b & b^2 \end{vmatrix} = 0.$$

d)

$$\begin{vmatrix} 1 & 3 & x \\ 3 & 1 & 5 \\ x & 2 & 10 \end{vmatrix} = 0.$$

5. Find the general equation of the straight-line AB and calculate the area of the triangle ABC

a) $A = [-1, 5]$, $B = [2, -6]$, $C = [4, 0]$.

b) $A = [-1, 18]$, $B = [1, 8]$, $C = [2, 3]$.

c) $A = [5, 0]$, $B = [0, 2]$, $C = [-2, -1]$.

6. Find the genearal equation of the plane ABC and calculate the volume of the tetrahedron $ABCD$

a) $A = [3, 0, 4]$, $B = [-1, -1, 7]$, $C = [0, -2, -3]$, $D = [6, 5, 4]$.

b) $A = [3, 4, 5]$, $B = [-2, -3, -4]$, $C = [6, 0, 8]$, $D = [3, 2, 7]$.