V. EIGENVALUES AND EIGENVECTORS

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Our objective now is to obtain an answer to the following question: under what conditions is an $n \times n$ matrix A similar to a diagonal matrix or a block matrix? We will consider $n \times n$ matrices A over $\mathbb{R}$ or $\mathbb{C}$.

Definition

The matrices A and B of size $n \times n$ are called *similar* matrices if there exists a regular matrix P such that

$$B = P^{-1} \cdot A \cdot P.$$

Definition

By an *eigenvalue of* A we mean a scalar λ for which there exists a *non-zero* vector such that

$$A \cdot x^t = \lambda \cdot x^t.$$

Such a (row) vector x is called an *eigenvector associated with* λ.

Observe that $Ax^t = \lambda x^t$ can be written as $(A - \lambda E)x^t = o^t$. Then λ is an eigenvalue of A, if and only if the system of equations $(A - \lambda E)x^t = o^t$ has a nontrivial solution. This is the case, if and only if matrix $A - \lambda E$ is not invertible, and this is equivalent to $\det(A - \lambda E) = 0$.

Theorem

Similar matrices have the same eigenvalues.

Definition

For a matrix $A = (a_{ij})_{n\times n}$ and a scalar λ we define the *characteristic matrix* of A by

$$A - \lambda E = \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{pmatrix}.$$

Definition

For $A = (a_{ij})_{n\times n}$ we call

$$\det(A-\lambda E) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \multicolumn{4}{c}{\dotfill} \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}$$

the *characteristic polynomial* of A.

Definition

Let $A = (a_{ij})_{n\times n}$. The equation

$$\det(A-\lambda E) = 0$$

is called the *characteristic equation* of A.

Note

The eigenvalues are thus the roots of the characteristic equation. Over the field $\mathbb{C}$ of complex numbers this equation has n roots, some of which may be repeated. If $\lambda_1, \dots, \lambda_k$ are the distinct roots (= eigenvalues), the characteristic polynomial factorizes to the form

$$(\lambda-\lambda_1)^{r_1}\cdot(\lambda-\lambda_2)^{r_2}\cdot\ldots\cdot(\lambda-\lambda_k)^{r_k}.$$

The polynomial is of degree n in λ. We call $r_1, \dots, r_k$ the *algebraic multiplicities* of $\lambda_1, \dots, \lambda_k$.

Examples

1. If

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 2 & 3 & -1 \end{pmatrix}, \quad \text{then} \quad A-\lambda E = \begin{pmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & -1 \\ 2 & 3 & -1-\lambda \end{pmatrix}$$

and the characteristic equation of A is

$$\det(A-\lambda E) = (1-\lambda)^2(-1-\lambda) - 4 + 0 - 0 - 0 + 3(1-\lambda) = -\lambda^3 + \lambda^2 - 2\lambda + 2 = 0.$$

2. If A is the matrix

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}, \quad \text{then} \quad A-\lambda E = \begin{pmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{pmatrix}$$

and the characteristic equation of the matrix A is

$$\det(A-\lambda E) = (2+\lambda)^2\cdot(4-\lambda) = 0,$$

so the eigenvalues are 4 and -2, the latter being of algebraic multiplicity 2.

Note

If λ is an eigenvalue of A, then the set

$$E_\lambda = \{x = (x_1, \dots, x_n) : Ax^t = \lambda x^t\},$$

i.e., the set of eigenvectors corresponding to λ together with the zero vector (column) o, is clearly a subspace. This subspace is called the *eigenspace associated with the eigenvalue* λ. The dimension of the eigenspace E_λ is called *the geometric multiplicity of the eigenvalue* λ.

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Let us continue in the preceding example. We have the matrix

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}, \quad \text{with} \quad \lambda_1 = 4, \quad \lambda_2 = \lambda_3 = -2.$$

To determine the eigenspace E_4, we have to solve $(A - 4E) \cdot x^t = o^t$, so that

$$\begin{pmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding system of equations yields $x = 0$, $y - z = 0$, hence E_4 is spanned by $x = (0, y, y)$, where $y \neq 0$, since eigenvectors are non-zero by definition. Consequently, we see that the eigenspace E_4 is of dimension 1 with the basis $\{(0, 1, 1)\}$.

We procede in a similar way to determine the eigenspace E_{-2}. We have to solve $(A + 2E) \cdot x^t = o^t$, thus

$$\begin{pmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding system of equations leads to to $x = y$, $z = 0$, therefore E_{-2} is spanned by $x = (x, x, 0)$, where $x \neq 0$. Consequently, we see that the eigenspace E_{-2} is of dimension 1 with the basis $\{(1, 1, 0)\}$.

Theorem

Eigenvectors that correspond to distinct eigenvalues are linearly independent.

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How to decide whether matrices A and B are similar?

1. Method using the definition

If A and B are matrices of size $n \times n$, then we try to find a regular matrix P of size $n \times n$, i.e., we solve the matrix equation $PA = BP$, so that we solve a system of n^2 linear equations with n^2 unknonws.

Example

Decide whether matrices

$$A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -10 & -4 \\ 26 & 11 \end{pmatrix}$$

are similar.

We want to find a matrix P of the form

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

where $x, y, z, t \in \mathbb{R}$ satisfying $PA = BP$. Hence

$$PA = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2x & x+3y \\ -2z & z+3t \end{pmatrix},$$

$$BP = \begin{pmatrix} -10 & -4 \\ 26 & 11 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} -10x-4z & -10y-4t \\ 26x+11z & 26y+11t \end{pmatrix}.$$

From the equality $PA = BP$ we obtain the following system of linear equations

$$\begin{aligned} -2x &= -10x - 4z, \\ x + 3y &= -10y - 4t, \\ -2z &= 26x + 11z, \\ z + 3t &= 26y + 11t. \end{aligned}$$

It is not difficult to find its solution: $\big(x, y, -2x, \frac{1}{4}(-x-13y)\big)$, where $x, y \in \mathbb{R}$. We can choose, for exemple $x = -1$, $y = 1$ and we obtain

$$P = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}.$$

So, we see that matrix A and B are similar.

This method is not reasonably fast and easy for matrices of size $n \times n$ with larger n.

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2. Reduction to Jordan canonical (normal) form

A short theoretical introduction

Definition

By an *elementary Jordan matrix* associated with $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$) we mean the square matrix of the form

$$\begin{pmatrix} \lambda & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & \lambda & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & 0 & \dots & 1 & \lambda \end{pmatrix},$$

in which the diagonal entries are all equal to λ, the entries immediately below the diagonal are all 1, and all other entries are 0.

Definition

By a *Jordan block matrix* associated with $\lambda_i \in \mathbb{R}$ (or $\lambda_i \in \mathbb{C}$) we mean a matrix J of the form

$$\begin{pmatrix} J_1 & & \\ & J_2 & \\ \dots & \dots & \dots \\ & & J_t \end{pmatrix},$$

where each J_i (the so called Jordan block) is an elementary Jordan matrix associated with λ_i.

Examples

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix A contains only one elementary Jordan matrix, matrix B contains two elementary Jordan matrices and matrix C contains three elementary Jordan matrices.

Notes

1. A Jordan block matrix is, strictly speaking, not unique since the order in which the Jordan blocks J_i appear along the diagonal is not specified. However, the number of such blocks, the size of each block, and the number of elementary Jordan matrices that appear in each block, are uniquely determined by A.
2. In a Jordan block matrix, the eigenvalue λ_i appears d_i times on the diagonal, and the number of elementary Jordan matrices associated with λ_i is the geometric multiplicity of the eigenvalue λ_i.
3. Two matrices are similar, if and only if they have the same system of Jordan blocks J_i.

Examples

1.

Matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

are similar.

2.

Matrices

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

are similar and their Jordan canonical matrix is the diagonal matrix.

3.

Matrices

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \text{are not similar.}$$

Definition

If A is similar to some Jordan block matrix J, we say that J is the so called *Jordan canonical matrix* or *Jordan normal matrix* of matrix A.

Definition

We say that an $n \times n$ matrix A is *diagonalizable*, if it is similar to a diagonal matrix D, thus if there exists an invertible matrix P such that $P^{-1} \cdot A \cdot P = D$.

Example

Matrices

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 2 \\ -2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

are similar, so that matrix B is the Jordan canonical form of the matrix A. Try to prove it.

Theorem

1. The matrix A is similar to the diagonal matrix D, if and only if its Jordan canonical matrix is diagonal.
2. An $n \times n$ matrix A is similar to a diagonal matrix D, if and only if A has n linearly independent eigenvectors. In this case the diagonal entries of D are the eingenvalues of A.
3. If matrices A and B are similar, then matrices A^t and B^t are similar, A^{-1} and B^{-1} are similar (if exist), A^k and B^k ($k \in \mathbb{N}$) are similar.

Note

We shall now consider the problem of finding such a matrix P. Firstly, we observe that the equation $P^{-1} \cdot A \cdot P = D$ can be written $AP = PD$. Let $p_1, \ldots, p_n$ be columns of P and let

$$D = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are the eingenvalues of A. Comparing the i-th columns of each side of the equation $AP = PD$, we obtain

$$A \cdot p_i = \lambda \cdot p_i, \quad i = 1, \ldots, n.$$

In other words, *the i-the column of P is an eigenvector of A correspondig to the eingenvalue λ_i.*

Now, we consider the problem of finding an invertible matrix P such that $P^{-1} \cdot A \cdot P = J$, when a matrix A is given.

There are four typical cases:

1. If each λ_j is an eigenvalue with the algebraic multiplicity 1, then for each λ_j there exists a 1-dimensional eigenspace. So in the j-the column of P, there is an eigenvector of A correspondig to the eigenvalue λ_j.
2. If λ_j is an eigenvalue with the algebraic multiplicity k and the equation $(A - \lambda_j E) \cdot x^t = o^t$ has k independent eigenvectors, then for that λ_j there exists a k-dimensional eigenspace. So in the j-the column of P, there is an eigenvector of A correspondig to the eigenvalue λ_j.
3. If λ_j is an eigenvalue with the algebraic multiplicity k and the equation $(A - \lambda_j E) \cdot x^t = o^t$ has only 1 independent eigenvector, then for that λ_j there exists a 1-dimensional eigenspace spanned by the vector v_1. We must find $k-1$ so called *generalized eigenvectors* $v_2, \ldots, v_k$, which are given by the following system

$$\begin{aligned} (A - \lambda_j E) \cdot v_1^t &= o^t, \\ (A - \lambda_j E) \cdot v_2^t &= v_1^t, \\ (A - \lambda_j E) \cdot v_3^t &= v_2^t, \\ &\ldots \\ (A - \lambda_j E) \cdot v_k^t &= v_{k-1}^t. \end{aligned}$$

So in the last column of P, there is an eigenvector v_1, in front of it there is a generalized eigenvector v_2, in front of it there is a generalized eigenvetor v_3, etc., and in the first column, there is a generalized eigenvector v_k.

4. If λ_j is an eigenvalue with the algebraic multiplicity k and the equation $(A - \lambda_j E) \cdot x^t = o^t$ has r independent eigenvectors, $1 < r < k$, then for that λ_j there exists an r-dimensional eigenspace. We must find r independent eigenvectors and $k - r$ generalized eigenvectors.

Examples

1. Consider

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 0 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

Find its Jordan canonical form.

I. The characteristic matrix is

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 2 & -2 \\ -1 & -\lambda & 2 \\ -2 & 2 & 1-\lambda \end{pmatrix}.$$

II. The corresponding polynomial is

$$\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & -\lambda & 2 \\ -2 & 2 & 1-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + 5\lambda - 6.$$

III. The eigenvalues are

$$(\lambda - 1)(\lambda + 2)(\lambda - 3) = 0 \quad \Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -2, \quad \lambda_3 = 3.$$

We obtain three different eigenvalues with multiciplity 1. The Jordan canonical form of the matrix A is

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

We see that the matrix A is diagonalizable. This case is easy.

2. Consider

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find its Jordan canonical form.

I. The characteristic matrix is

$$A - \lambda E = \begin{pmatrix} 3-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix}.$$

II. The corresponding polynomial is

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = -(\lambda-2)^3.$$

III. The eigenvalues are

$$-(\lambda-2)^3 = 0 \quad \Rightarrow \quad \lambda_{1,2,3} = 2.$$

We obtain only one eigenvalue with multiciplity 3. To obtain the Jordan canonical form of the matrix A we must know the dimension of the vector space which is formed by solutions of the equation $(A-2E)x^t = o^t$, thus

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As we see, the dimension of the vector space is 2 (the rank of the matrix $(A-2E)$ is 1). We have one eigenvalue with multicipility 3 and two independent eigenvectors, therefore the Jordan canonical form of the matrix A is composed of two Jordan block matrices

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

We see that the matrix A is not diagonalizable.

3. Consider

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Find its Jordan canonical form.

I. The characteristic matrix is

$$A - \lambda E = \begin{pmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{pmatrix}.$$

II. The corresponding polynomial is

$$\begin{vmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = -\lambda \cdot (\lambda+3)^2.$$

III. The eigenvalues are

$$-\lambda(\lambda+3)^2 = 0 \quad \Rightarrow \quad \lambda_1 = 0, \quad \lambda_{2,3} = -3.$$

We obtain one eigenvalue with multiciplity 1 and one eigenvalue with multiplicity 2. To obtain the Jordan canonical form of the matrix A we must determine the dimension of the vector space formed by solutions of the equation $(A+3E)x^t = o^t$, hence

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As we see, the dimension of the vector space is 2. We have one eigenvalue with multipicity 2 and two independent eigenvectors (for $\lambda = -3$) and one eigenvalue with multiplicity 1 (for $\lambda = 0$). The Jordan canonical form of the matrix A is composed of tre Jordan block matrices (one for $\lambda = 0$ and two for $\lambda = -3$).

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see that the matrix A is diagonalizable.

4. Consider

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Find its Jordan canonical form and the corresponding matrix P.

I. The characteristic matrix is

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix}.$$

II. The corresponding polynomial is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (\lambda-1)^2 \cdot (2-\lambda).$$

III. The eigenvalues are

$$(\lambda-1)^2\cdot(2-\lambda)=0 \quad\Rightarrow\quad \lambda_1=2,\quad \lambda_{2,3}=1.$$

We obtain one eigenvalue with multiciplity 1 and one eigenvalue with multiplicity 2. To obtain the Jordan canonical form of the matrix A we must determine the dimension of the vector space which is formed by solutions of the equation $(A-E)x^t=o^t$, thus

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As we see, the dimension of this vector space is 1, i.e., we have one eigenvalue with multipicity 2 and one eigenvector and one eigenvalue with multiplicity 1. The Jordan canonical form of the matrix A is composed of two Jordan block matrices

$$J=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\quad\text{or}\quad J=\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Therefore the matrix A is not diagonalizable.

IV. Now, we will find the matrix P. Firstly, we find the vector space which is formed by solutions of the equation $(A-2E)x^t=o^t$, that is,

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the vector space is spanned by the set $\{(2,1,1)\}$.

V. Now, we find the vector space which is formed by solutions of the equation $(A-E)x^t=o^t$, that is,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the vector space is spanned by the set $\{(1,0,0)\}$.

VI. Now, we find the generalized eigenvector which is formed by solutions of the equation $(A-E)x^t=(1,0,0)^t$, hence

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the generalized eigenvector is, for example, $(0,1,0)$.

The the matrix P is

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. Consider

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find its Jordan canonical form and matrix P.

I. The characteristic matrix is

$$A - \lambda E = \begin{pmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix}.$$

II. The corresponding polynomial is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3.$$

III. The eigenvalues are

$$(1-\lambda)^3 = 0 \quad \Rightarrow \quad \lambda_{1,2,3} = 1.$$

We obtain one eigenvalue with multiciplity 3. To obtain the Jordan canonical form of the matrix A we must determine the dimension of the vector space which is formed by solutions of the equation $(A-E)x^t = o^t$, i.e.,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As we see the dimension of the vector space is 1. We have one eigenvalue with multipicity 3 and one eigenvector. The Jordan canonical form of the matrix A is composed of one Jordan block matrix.

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We see that the matrix A is not diagonalizable.

IV. Now, we will find the matrix P. At first we determine the vector space formed by solutions of the equation $(A-E)x^t = o^t$, hence

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the vector space is spanned by the set $\{(1,0,0)\}$.

V. Now, we find the first generalized eigenvector which is a solution of the equation $(A-E)x^t = (1,0,0)^t$, that is,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the first generalized eigenvector is, for example, $(0,1,0)$.

VI. Now, we find the second generalized eigenvector which is a solution of the equation $(A-E)x^t = (0,1,0)^t$, i.e.,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

We see that, for example, $(0,-1,1)$ can be taken as the second generalized eigenvector.

The the matrix P is

$$\begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

7. Consider

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Find its Jordan canonical form and matrix P.

I. The characteristic matrix is

$$A - \lambda E = \begin{pmatrix} 3-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix}.$$

II. The corresponding polynomial is

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)^3.$$

III. The eigenvalues are

$$(2-\lambda)^3 = 0 \quad \Rightarrow \quad \lambda_{1,2,3} = 2.$$

We obtain one eigenvalue with multiciplity 3. To obtain the Jordan canonical form of the matrix A we must determine the dimension of the vector space which is formed by solutions of the equation $(A-2E)x^t = o^t$, that is

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

As we see, the dimension of the vector space is 2, i.e., we have one eigenvalue with multipicity 3 and two linearly independent eigenvectors. The Jordan canonical form of the matrix A is composed of two Jordan block matrices.

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{or} \quad J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

We see the that matrix A is not diagonalizable.

IV. Now we will find the matrix P. We will find the vector space which is the solution of the equation $(A-2E)x^t = o^t$, thus

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the vector space is spanned by the set $\{(0,1,1),(1,0,1)\}$.

V. Now, we try to find the first generalized eigenvector which is a solution of the equation $(A-2E)x^t = (0,1,1)^t$, thus

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We see that the equation has no solution. It does not matter because we can use the second eigenvector $(1,0,1)$.

VI. Now, we try to find the first generalized eigenvector which is a solution of the equation $(A-2E)x^t = (1,0,1)^t$, hence

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

We see that the first generalized eigenvector may be chosen, for example, as $(1,0,0)$.

The matrix P is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

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Note

The Jordan canonical form is very useful for solving systems of linear differential equations. It is not our intention to discuss this subject here because it heavily depends on the theory. It will be illustrated in the second semester in the course *Calculus 2*.

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Exercises

1. Calculate eigenvalues and eigenvectors of the following matrices.

a)

$$\begin{pmatrix} 2 & 5 \\ -2 & 3 \end{pmatrix}.$$

b)

$$\begin{pmatrix} 1 & -3 \\ -4 & 6 \end{pmatrix}.$$

c)

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

d)

$$\begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}.$$

e)

$$\begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}.$$

f)

$$\begin{pmatrix} 2 & 0 & 0 \\ -5 & 7 & -5 \\ -10 & 10 & -8 \end{pmatrix}.$$

g)

$$\begin{pmatrix} 2 & 5 & -6 \\ 4 & 6 & -9 \\ 3 & 6 & -8 \end{pmatrix}.$$

2. Calculate eigenvalues and eigenvectors of the following matrices.

a)

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

b)

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}.$$

c)

$$\begin{pmatrix} -9 & 8 & -3 \\ -19 & 17 & -7 \\ -12 & 10 & -4 \end{pmatrix}.$$

d)

$$\begin{pmatrix} -3 & -12 & 24 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}.$$

e)

$$\begin{pmatrix} -2 & 1 & -2 \\ -2 & 4 & 2 \\ 7 & -1 & 7 \end{pmatrix}.$$

g)

$$\begin{pmatrix} 0 & 3 & -2 \\ -2 & 5 & -1 \\ 1 & -1 & 4 \end{pmatrix}.$$

h)

$$\begin{pmatrix} 1 & -2 & 0 \\ 1 & 4 & 0 \\ 1 & 2 & 2 \end{pmatrix}.$$

i)

$$\begin{pmatrix} -2 & 3 & -3 \\ -6 & 7 & -6 \\ -6 & 6 & -5 \end{pmatrix}.$$

j)

$$\begin{pmatrix} 1 & -9 & 15 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}.$$

k)

$$\begin{pmatrix} 3 & 1 & -2 \\ 0 & 5 & -4 \\ -2 & 3 & -1 \end{pmatrix}.$$

3. Calculate Jordan canonical form of the matrices.

a)

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

b)

$$\begin{pmatrix} 2 & 5 \\ 5 & 3 \end{pmatrix}.$$

c)

$$\begin{pmatrix} 1 & 0 \\ -4 & 6 \end{pmatrix}.$$

d)

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}.$$

e)

$$\begin{pmatrix} 2 & 7 \\ -1 & 2 \end{pmatrix}.$$

4. Calculate Jordan canonical form of the matrices.

a)

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

b)

$$\begin{pmatrix} 5 & -2 & 1 \\ 5 & -1 & 1 \\ 2 & -1 & 2 \end{pmatrix}.$$

c)

$$\begin{pmatrix} -2 & -1 & 1 \\ 5 & -1 & 4 \\ 5 & 1 & 2 \end{pmatrix}.$$

d)

$$\begin{pmatrix} 9 & -6 & -2 \\ 18 & -12 & -3 \\ 18 & -9 & -6 \end{pmatrix}.$$

e)

$$\begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}.$$

f)

$$\begin{pmatrix} 3 & -1 & 0 \\ 6 & -3 & 2 \\ 8 & -6 & 5 \end{pmatrix}.$$

g)

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

h)

$$\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}.$$

i)

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

j)

$$\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}.$$

k)

$$\begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}.$$

l)

$$\begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}.$$

m)

$$\begin{pmatrix} 3 & -1 & 0 \\ 6 & -3 & 2 \\ 8 & -6 & 5 \end{pmatrix}.$$

n)
$$\begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}.$$

o)
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

p)
$$\begin{pmatrix} 8 & 15 & -36 \\ 8 & 21 & -46 \\ 5 & 12 & -27 \end{pmatrix}.$$

q)
$$\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}.$$

r)
$$\begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}.$$

s)
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 9 & 0 \end{pmatrix}.$$

t)
$$\begin{pmatrix} 5 & 2 & -3 \\ 4 & 5 & -4 \\ 6 & 4 & -4 \end{pmatrix}.$$

u)
$$\begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}.$$

v)
$$\begin{pmatrix} 0 & -4 & 0 \\ 1 & -4 & 0 \\ 1 & -2 & -2 \end{pmatrix}.$$

5. Calculate Jordan canonical form of the matrix A and determine, if possible, a matrix P satisfying $J = P^{-1} \cdot A \cdot P$.

a)
$$\begin{pmatrix} 12 & -6 & -2 \\ 18 & -9 & -3 \\ 18 & -9 & -3 \end{pmatrix}.$$

b)

$$\begin{pmatrix} 3 & 1 & -1 \\ -3 & -1 & 3 \\ -2 & -2 & 4 \end{pmatrix}.$$

c)

$$\begin{pmatrix} 6 & 0 & 8 \\ 3 & 2 & 6 \\ -2 & 0 & -2 \end{pmatrix}.$$

d)

$$\begin{pmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{pmatrix}.$$

e)

$$\begin{pmatrix} 24 & -11 & -22 \\ 20 & -8 & -20 \\ 12 & -6 & -10 \end{pmatrix}.$$

f)

$$\begin{pmatrix} 3 & 2 & -5 \\ 2 & 6 & -10 \\ 1 & 2 & -3 \end{pmatrix}.$$

g)

$$\begin{pmatrix} 6 & 20 & -34 \\ 6 & 32 & -51 \\ 4 & 20 & -32 \end{pmatrix}.$$

h)

$$\begin{pmatrix} 6 & 6 & -15 \\ 1 & 5 & -5 \\ 1 & 2 & -2 \end{pmatrix}.$$

i)

$$\begin{pmatrix} 37 & -20 & -4 \\ 34 & -17 & -4 \\ 119 & -70 & -11 \end{pmatrix}.$$

j)

$$\begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}.$$

k)

$$\begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}.$$

l)

$$\begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}.$$