III. SYSTEMS OF LINEAR EQUATIONS

Martina Bečvářová

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Faculty of Transportation Sciences, CTU in Prague

Repetition of knowledge from a secondary school

We shall now consider in detail a systematic method of solving systems of linear equations. In working with such a system, there are three basic operations involved, namely

(1) interchanging of equations (usually for convenience);
(2) multiplying an equation by a non-zero scalar;
(3) forming a new equation by adding one equation to another.

Example

Solve the system

$$\begin{aligned} y + 2z &= 1, \\ x - 2y + z &= 0, \\ 3y - 4z &= 23. \end{aligned}$$

We mupltiply the first equation by 3 and substract the new equation from the third equation to obtain

$$-10z = 20,$$

whence $z = -2$. It follows from the first equation that $y = 5$, and then from the second equation that $x = 2y - z = 10 - (-2) = 12$. So the solution is $[x, y, z] = [12, 5, -2]$.

The above example was chosen to raise the question: is there a *systematic* method of tackling systems of linear equations avoiding the haphazard manipulation of the equations that will yield all solutions when they exist, and make it clear when no solution is possible?

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Definition

By a *system of m linear equations with n unknown* we shall mean the following system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

where $x = (x_1, x_2, \dots, x_n)$ is a vector of unknowns (from $\mathbb{R}$ or $\mathbb{C}$), $a_{11}, a_{12}, \dots, a_{mn}$ are the coefficients (from $\mathbb{R}$ or $\mathbb{C}$), $b = (b_1, b_2, \dots, b_m)$ (from $\mathbb{R}$ or $\mathbb{C}$) and b_i are called the *right hand sides* of the system of linear equations.

Note

We will write

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \qquad x = (x_1, x_2, \dots, x_n), \qquad b^t = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix},$$

$$(A|b) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & | & b_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & | & b_m \end{pmatrix}.$$

The matrix A is called a *matrix of the system*, the matrix $(A|b)$ is called an *augmented matrix.*

For better undestanding we will write a system of linear equations as

$$Ax^t = b^t \qquad \text{or} \qquad \sum_{j=1}^{n} a_{ij}x_j = b_i \quad \text{for } i = 1, \dots, m \qquad \text{or}$$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \cdot (x_1, x_2, \dots, x_n)^t = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}.$$

The system is completely determined by its augmented matrix. In order to work solely with this, we consider the following *elementary row operations* on this matrix

(1) interchange two rows;
(2) multiply a row by a non-zero scalar;
(3) add one row to another.

These elementary row operations *do not affect the solutions (if any) of the system.* In fact, if the original system of equations has a solution, then this solution is also a solution of the system obtained by applying any of (1), (2), (3) and conversely.

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Definition

The system $Ax^t = o^t$, so that is $b = o = (0, 0, \ldots, 0)$, is called *homogeneous system of linear equations.* The system $Ax^t = b^t$, where $b \neq (0, 0, \ldots, 0)$, is called the *non-homogeneous system of linear equations.*

Definition

We shall say that a system of linear equations is *consistent*, if it has a solution. Otherwise, we shall say that it is *inconsistent.*

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Now we will apply our knowledge on matrices to solve system of linear equations to obtain a simple sytematic method of finding a solution.

There are three main problems

1. How to formulate necessary and sufficient conditions for the existence of a solution of the system of linear equations?
2. How the structure of solutions looks like, hence how to describe effectively and simply all solutions?
3. How to obtain an effective methods of solving a system of linear equations.

Theorem – a necessary and sufficient condition for the existence of solution
(so called Frobenius' theorem)

A non-homogeneous system $Ax^t = b^t$ has a solution, if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix.

We can write $\operatorname{rank} A = \operatorname{rank}(A|b)$.

The theorem easily follows from the following diagram equality

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{mn} \end{pmatrix} x_n = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}.$$

Note

a) always $\operatorname{rank} A \leq \operatorname{rank}(A|b)$,
b) if $\operatorname{rank} A = \operatorname{rank}(A|b)$, then there exists a solution,
c) if $\operatorname{rank} A < \operatorname{rank}(A|b)$, then no solution exists.

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Homogeneous system of linear equations

From the theorem stated above it is seen that the system $Ax^t = o^t$ has always the so called trivial solution $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$.

When does a non-trivial solution exist?

Theorem

If A is an $m \times n$ matrix, then the homogeneous system $Ax^t = o^t$ has a non-trivial solution, if and only if the rank of A is less than n.

In other words, a non-trivial solution exists, if and only if the columns of A are linearly dependent. Since A has n columns, this is the case, if and only if the (column) rank of A is less than n.

Note

All solutions of the system of linear equations form a vector space V. Its dimension is $\dim V = n - \operatorname{rank} A$, where n is the number of uknowns. So we must find $n - \operatorname{rank} A$ linearly independent solutions which form the basis of the vector space V.

Note

a) Always $\operatorname{rank} A \leq \operatorname{rank}(A|b)$,
b) if $\operatorname{rank} A = n$, then only trivial solution exists,
c) if $\operatorname{rank} A < n$, then a non-trivial solution exists.

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Non-homogeneous system of linear equations

We know when a solution of a non-homogeneous system of linear equations exists. But how all solutions look like?

Theorem

Let A is a $m \times n$ coefficient matrix of a consistent system of linear equations. If the rank of A is p then $n - p$ of unknowns can be chosen as parameters and a new system can be considered.

Note

In other words: every solution of the system $Ax^t = b^t$ can be expressed as a sum of one *particular* solution of the non-homogeneous system and a linear combination of all solutions of homogeneous system with the same matrix A.

a) Always $\operatorname{rank} A \leq \operatorname{rank}(A|b)$,
b) if $\operatorname{rank} A = n$, then there exists exactly one solution,
c) if $\operatorname{rank} A < n$, then $n - \operatorname{rank} A$ independent solutions of a homogeneous system exist.

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Gauss' algorithm for solving a system of linear equations

We will use the results explained in Chapter II. As we know: *Every non-zero matrix can be transformed by means of elementary row operations to a row-echelon matrix.* So working with the augmented matrix $(A|b)$, or simply with A in the homogeneous case, we perform row operations to transform $(A|b)$ and A we will obtain a row-echelon matrix. We will learn everything from a row-echelon matrix what is needed for solving a system of linear equations.

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Examples – homogeneous system of linear equations

1. Solve the following system of linear equations

$$\begin{aligned} x + y + z &= 0, \\ 2x - y + 2z &= 0, \\ x - 2y + 2z &= 0. \end{aligned}$$

We have

$$\begin{pmatrix} (x & y & z) \\ 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, the following three equations follow from the last matrix:

$$1 \cdot z = 0, \quad -3 \cdot y = 0, \quad 1 \cdot x + 1 \cdot y + 1 \cdot z = 0,$$

thus $z = 0$, $y = 0$ and $x = y - z = 0$. Our system has the trivial solution $\{(0, 0, 0)\}$ only.

2. Solve the following system of linear equations

$$\begin{aligned} x + y + z &= 0, \\ 2x - y + 2z &= 0, \\ x - 2y + z &= 0. \end{aligned}$$

We get

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ 1 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix A is 2, our system has 3 unknowns, so the vector space formed by the solutions of our system has dimension $\dim V = 3 - 2 = 1$. Now we can proceed in the reversed order. We see that the next two equations follow from the last matrix

$$-3 \cdot y = 0, \quad x + y + z = 0.$$

So $y = 0$, $x = -y - z = 0 - z = -z$. We can choose an arbitrary $z \in \mathbb{R}$. The solutions of our system are $\{(-z, 0, z)\}, z \in \mathbb{R}$.

We can write $\{z \cdot (-1, 0, 1), z \in \mathbb{R}\}$ or better we can use the style from Chapter II. and we can use the notion of a vector space. We prefer to put

$$B = \{(-1, 0, 1)\},$$

and express the set of all solutions using the basis of the vector space.

3. Solve the following system of linear equations

$$\begin{aligned} x + y + z + t &= 0, \\ 2x - y + z &= 0. \end{aligned}$$

Here

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & -1 & -2 \end{pmatrix}.$$

The rank of matrix A is 2, our system has 4 unknowns, so the vector space consisting of solutions of our system has dimension $\dim V = 4 - 2 = 2$. We must find two linearly independent solutions. Now we will use the last matrix. The next two equations follow from the last matrix

$$-3y - 1z - 2t = 0, \quad x + y + z + t = 0.$$

We can choose, for example, unkowns z and t as parameters, so that $B = \{(*, *, 1, 0), (*, *, 0, 1)\}$.

From the first equation we can calculate $y = -\frac{1}{3}(z + 2t)$, thus $B = \{(*, -\frac{1}{3}, 1, 0), (*, -\frac{2}{3}, 0, 1)\}$.

Now from the second equation we can calculate $x = -y - z - t$, hence

$$B = \{(-\frac{2}{3}, -\frac{1}{3}, 1, 0), (-\frac{1}{3}, -\frac{2}{3}, 0, 1)\}.$$

Consequently, every solution can be written as

$$u = z(-\frac{2}{3}, -\frac{1}{3}, 1, 0) + t(-\frac{1}{3}, -\frac{2}{3}, 0, 1), \quad \text{where } z, t \in \mathbb{R}.$$

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Examples – non-homogeneous system of linear equations

1. Solve the following system of linear equations

$$\begin{aligned} x + y + z &= 1, \\ 2x - y + 2z &= 1, \\ 3x + z &= 2. \end{aligned}$$

Here

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 2 & -1 & 2 & | & 1 \\ 3 & 0 & 1 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 0 & | & -1 \\ 0 & -3 & -2 & | & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -3 & 0 & | & -1 \\ 0 & 0 & -2 & | & 0 \end{pmatrix}.$$

The rank of the matrix A is 3, the rank of the augmented matrix $(A|b)$ is 3, our system has 3 unknowns, so there exists exactly one solution. As we see, the following three equations from the last matrix are

$$-2z = 0, \quad -3y = -1, \quad x + y + z = 1.$$

We get $z = 0$, $y = \frac{1}{3}$, $x = 1 - y - z = 1 - \frac{1}{3} - 0 = \frac{2}{3}$, so our solution is $[\frac{2}{3}, \frac{1}{3}, 0]$.

2. Solve the following system of linear equations

$$\begin{aligned} x + y + z &= 1, \\ x - y + z &= 2, \\ x + z &= 3. \end{aligned}$$

We have

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & -1 & 1 & | & 2 \\ 1 & 0 & 1 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -2 & 0 & | & 1 \\ 0 & -1 & 0 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -2 & 0 & | & 1 \\ 0 & 0 & 0 & | & -3 \end{pmatrix}.$$

The rank of the matrix A is 2, the rank of the augmented matrix $(A|b)$ is 3; since $2 \neq 3$, our system of linear equations has no solution. (From the last matrix we obtain the equation $0x + 0y + 0z = -3$ which obviously has no solution.)

3. Solve the following system of linear equations

$$\begin{aligned} y + z &= 1, \\ x - 2y + z &= 2, \\ x - y + 2z &= 3. \end{aligned}$$

We have

$$\begin{pmatrix} 0 & 1 & 1 & | & 1 \\ 1 & -2 & 1 & | & 2 \\ 1 & -1 & 2 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 1 & 1 & | & 1 \\ 0 & 1 & 1 & | & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

The rank of the matrix A is 2, the rank of the augmented matrix $(A|b)$ is 2, our system has 3 unknowns. So every solution is of the form $u + V$, where u is a particular solution of non-homogeneous system and V

is the vector space consisting of all solutions of the homogeneous system. The following equations follow from the last matrix

$$y + z = 1, \quad x - 2y + z = 2.$$

a) Now we find one particular solution of the non-homogeneous system of linear equations corresponding to the augmented matrix

$$\begin{pmatrix} 1 & -2 & 1 & | & 2 \\ 0 & 1 & 1 & | & 1 \end{pmatrix}.$$

We can choose one unknown as a parameter, for example $z = 0$, hence $[*, *, 0]$. From the first equation $y = 1 - z = 1$, so that $[*, 1, 0]$. From the second equation $x = 2 + 2y - z = 2 + 2 - 0 = 4$, so that $[4, 1, 0]$.

b) Now we find all solutions of the homogeneous system with matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We can choose one unknown as a parameter, for example $z = 1$, so

$$B = \{(*, *, 1)\}.$$

We obtain from the first equation $y = -z = -1$, hence $B = \{(*, -1, 1)\}$. The second equation yields $x = 2y - z = -2 - 1 = -3$, therefore $B = \{(-3, -1, 1)\}$.

c) All solutions of our system are given by

$$[4, 1, 0] + \{(-3, -1, 1)\}.$$

4. Solve the following system of linear equations

$$\begin{aligned} x - y + z + u - 2v &= 0, \\ 2x + y - z - u + v &= 1, \\ 3x + 3y - 3z - 3u + 4v &= 2, \\ 4x + 5y - 5z - 5u + 7v &= 3. \end{aligned}$$

We have

$$\begin{pmatrix} 1 & -1 & 1 & 1 & -2 & | & 0 \\ 2 & 1 & -1 & -1 & 1 & | & 1 \\ 3 & 3 & -3 & -3 & 4 & | & 2 \\ 4 & 5 & -5 & -5 & 7 & | & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 1 & -2 & | & 0 \\ 0 & 3 & -3 & -3 & 5 & | & 1 \\ 0 & 6 & -6 & -6 & 10 & | & 2 \\ 0 & 9 & -9 & -9 & 15 & | & 3 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & -1 & 1 & 1 & -2 & | & 0 \\ 0 & 3 & -3 & -3 & 5 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 1 & -2 & | & 0 \\ 0 & 3 & -3 & -3 & 5 & | & 1 \end{pmatrix}.$$

The rank of the matrix A is 2, the rank of the augmented matrix $(A|b)$ is 2, our system has 5 unknowns. So every solution has the form $u + V$, where u is a particular solution of non-homogeneous system and V is the vector space of dimension 3.

a) Now we find a particular solution of the non-homogeneous system corresponding to

$$\begin{pmatrix} 1 & -1 & 1 & 1 & -2 & | & 0 \\ 0 & 3 & -3 & -3 & 5 & | & 1 \end{pmatrix}.$$

We can choose three unknowns as a parameters, for example $z = 0$, $u = 0$, $v = 0$, thus $[*, *, 0, 0, 0]$. From the first equation we get $3y = 1$, so that $[*, \frac{1}{3}, 0, 0, 0]$. From the second equation, $x = \frac{1}{3}$, which gives $\left[\frac{1}{3}, \frac{1}{3}, 0, 0, 0\right]$.

b) Now we find all solutions of the homogeneous system with matrix

$$\begin{pmatrix} 1 & -1 & 1 & 1 & -2 \\ 0 & 3 & -3 & -3 & 5 \end{pmatrix}.$$

We can choose three unknowns as parameters, for example z, u, v, hence

$$B = \{(*, *, 0, 0, 1), (*, *, 0, 1, 0), (*, *, 1, 0, 0)\}.$$

From the equations $x - y + z + u - 2v = 0$ and $3y - 3z - 3u + 5v = 0$ we obtain three linearly independent vectors

$$B = \left\{\left(\frac{1}{3}, -\frac{5}{3}, 0, 0, 1\right), (0, 1, 0, 1, 0), (0, 1, 1, 0, 0)\right\}.$$

c) All solutions of our system are given by

$$\left[\frac{1}{3}, \frac{1}{3}, 0, 0, 0\right] + \left\{\left(\frac{1}{3}, -\frac{5}{3}, 0, 0, 1\right), (0, 1, 0, 1, 0), (0, 1, 1, 0, 0)\right\}.$$

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Systems of linear equations with a regular matrix

Theorem

Let $Ax^t = b^t$ be a non-homogeneous system of linear equations where A is a regular matrix $n \times n$. Then the system has only one solution

$$x^t = A^{-1}b^t,$$

where A^{-1} is an inverse matrix of matrix A.

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Matrix equations

Definition

Let A be a matrix of size $m \times n$ and B a matrix of size $m \times p$. By a matrix equation we shall mean the equation of the form

$$A \cdot X = B, \quad \text{where} \quad X \quad \text{is a matrix of size} \quad n \times p.$$

Theorem
The equation $A \cdot X = B$ has a solution, if and only if the rank of matrix A is equal to the rank of matrix $(A|B)$.

Note

There are three possibilities

1. If $A \cdot X = B$, where A is a regular matrix, then the equation has only one solution $X = A^{-1} \cdot B$.
2. If $A \cdot X = B$, where A is not a regular matrix and $\operatorname{rank} A < \operatorname{rank}(A|B)$, then the equation has no solution.
3. If $A \cdot X = B$, where A is not a regular matrix and $\operatorname{rank} A = \operatorname{rank}(A|B)$, then the equation has more than one solution. All solutions form a vector space of dimension $p \cdot (n - \operatorname{rank} A)$.

Exercises

Find matrices X for which

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \cdot X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Exercises

1. Solve the system of linear equations

a)

$$\begin{aligned} 2x - y - z &= 4, \\ 3x + 4y - 2z &= 2, \\ 3x - 2y + 4z &= 11. \end{aligned}$$

b)

$$\begin{aligned} (i+1)x + (1-i)y + (1+i)z &= 1, \\ (1-i)x + (1+3i)y + (i-1)z &= 0, \\ x + (1+i)y + iz &= 1. \end{aligned}$$

c)

$$\begin{aligned} 2x + (2+2i)y + \quad 2iz &= 1, \\ (1-i)x + (1+3i)y + (i-1)z &= 0, \\ (1+i)x + \ (1-i)y + (1+i)z &= 1. \end{aligned}$$

d)

$$\begin{aligned} x + \qquad\qquad\quad 2iz &= i, \\ ix + (2-i)y + (i+1)z &= 1+i, \\ (1-i)x + \quad iy - \quad\quad z &= 1. \end{aligned}$$

e)

$$\begin{aligned} 4x + 3y + 2z &= 1, \\ x + 3y + 5z &= 1, \\ 3x + 6y + 9z &= 2. \end{aligned}$$

f)

$$\begin{aligned} 2x - \ \ y + 3z &= 0, \\ x + \ 3y + 2z &= 0, \\ 3x - \ 5y + 4z &= 0, \\ x + 17y + 4z &= 0. \end{aligned}$$

g)

$$\begin{aligned} x + 3y + \ z &= 5, \\ 2x + \ y + \ z &= 2, \\ x + \ y + 5z &= -7. \end{aligned}$$

h)

$$\begin{aligned} x + \ y - 3z &= -1, \\ 2x + \ y - 2z &= 1, \\ x + \ y + \ z &= 3, \\ x + 2y - 3z &= 1. \end{aligned}$$

i)

$$\begin{aligned} x + \ 3y + 2z &= 2, \\ 2x - \ \ y + 3z &= 7, \\ 3x - \ 5y + 4z &= 12, \\ x + 17y + 4z &= -4. \end{aligned}$$

j)

$$\begin{aligned} x+y+z+u &= 0, \\ x+2y+3z+u &= 0, \\ x+3y+5z+7u &= 2, \\ x+4y+7z+10u &= 0. \end{aligned}$$

k)

$$\begin{aligned} x+2y-z+u &= 1, \\ 2x-y+4z+10u &= 2, \\ x+3z-5u &= 15, \\ 2x+5y+2z+2u &= 16. \end{aligned}$$

l)

$$\begin{aligned} 2x+3y-z+2u &= 3, \\ 5x+7y-4z+7u &= 8, \\ x+2y+z-u &= 1, \\ 4x+7y+z &= 5. \end{aligned}$$

m)

$$\begin{aligned} x+2y+3z-u &= 0, \\ x+5y+5z-4u &= -4, \\ x-y+z+2u &= 4, \\ x+8y+7z-7u &= 6. \end{aligned}$$

n)

$$\begin{aligned} x+2y &= -1, \\ y+z &= 0, \\ x-u &= -1, \\ x+y-z+u &= 2. \end{aligned}$$

o)

$$\begin{aligned} x+3y+5z+7u &= 12, \\ 3x+5y+7z+u &= 0, \\ 5x+7y+z+3u &= 4, \\ 7x+y+3z+5u &= 16. \end{aligned}$$

p)

$$\begin{aligned} x-\ 2y+3z-4u &= 4,\\ y-\ z+\ u &= -3,\\ x+\ 3y-\ \ 3u &= 1,\\ -7y+3z+\ u &= -3. \end{aligned}$$

q)

$$\begin{aligned} 2y+3z+4u &= 0,\\ x+\ \ 3z+4u &= -1,\\ x+2y+\ \ 4u &= -1,\\ x+2y+3z\ \ &= -1. \end{aligned}$$

r)

$$\begin{aligned} x+3y+5z+7u &= 12,\\ 3x+5y\ \ 7z+\ u &= 0,\\ 5x+7y+\ z+3u &= 4,\\ 7x+\ y+3z+5u &= -1. \end{aligned}$$

s)

$$\begin{aligned} 3x+\ y+2z-\ u &= 2,\\ 2x+3y-\ z+3u &= -1,\\ 4x+2y+2z+\ u &= 3,\\ x+2y-\ z+\ u &= 1. \end{aligned}$$

t)

$$\begin{aligned} x-2y+3z-4u &= 4,\\ y-\ z+\ u &= -3,\\ x+3y-\ \ 3u &= 1,\\ -\ 7y+3z+\ u &= -3. \end{aligned}$$

u)

$$\begin{aligned} 2x+\ y-\ z-\ u &= 4,\\ x+\ y+\ z+\ u &= 2,\\ x+2y+3z+4u &= 7,\\ 3x+2y-7z+2u &= 13. \end{aligned}$$

v)

$$\begin{aligned}
2x+3y+\quad 5u &= 2,\\
x+\ y+5z+2u &= 1,\\
2x+\ y+3z+2u &= 3,\\
x+\ y+3z+4u &= 3.
\end{aligned}$$

2. Solve the system of linear equations

a)

$$\begin{aligned}
x+2y\qquad &= -1,\\
y+z\quad &= 0,\\
x-\qquad u &= -1,\\
x+\ y-z+u &= 2.
\end{aligned}$$

b)

$$\begin{aligned}
x+3y+z-\ u-3v\ &= 2,\\
4x-\ y+z-3u-\ v\ &= 4,\\
-7x+5y-z+\ u-\ v\ &= -5,\\
x-5y-z-\ u+\ vv &= -2.
\end{aligned}$$

c)

$$\begin{aligned}
3x+\ y-2z+\ u-\ v &= 1,\\
2x-\ y+7z-3u+5v &= 2,\\
x+3y-2z+5u-7v &= 3,\\
3x-2y+7z-5u+8v &= 3.
\end{aligned}$$

d)

$$\begin{aligned}
2x-\ 2y+2z-\ u+\quad v &= 1,\\
x+\ 2y-\ z+\ u-\ 2v &= 1,\\
4x-10y+5z-5u+\ 7v &= 1,\\
2x+14y+7z-7u+11v &= -1.
\end{aligned}$$

e)

$$\begin{aligned}
2x+\ y+\ z+\ u+\ v &= 2,\\
x+2y+\ z+\ u+\ v &= 0,\\
x+\ y+3z+\ u+\ v &= 0,\\
x+\ y+\ z+4u+\ v &= -2,\\
x+\ y+\ z+\ u+5v &= 5.
\end{aligned}$$

f)

$$\begin{aligned}
x+2y-3z+4u-v &= -1,\\
2x-y+3z-4u+2v &= 8,\\
3x+y-z+2u-v &= 3,\\
4x+3y+4z+2u+2v &= -2,\\
x-y-z+2u-3v &= -3.
\end{aligned}$$

g)

$$\begin{aligned}
x-y+z+u-2v &= 0,\\
2x+y-z-u+v &= 1,\\
3x+3y-3z-3u+4v &= 2,\\
4x+5y-5z-5u+7v &= 3.
\end{aligned}$$

h)

$$\begin{aligned}
y+2z+2u+v &= 2,\\
x+z+2t+u+2v &= 1,\\
2x+y+z+t &= 2,\\
y+2z &= 0.
\end{aligned}$$

i)

$$\begin{aligned}
y+2z+2t+2v &= 0,\\
x+2y+z+2t+u &= 0,\\
y+2z+t+2u &= 1,\\
2x+2y+z+t+v &= 0.
\end{aligned}$$

j)

$$\begin{aligned}
ix+y &= -1,\\
3x+3y-z &= 0,\\
2x-y-2z &= -4+3i.
\end{aligned}$$

k)

$$\begin{aligned}
2x+y+4z+t &= 1,\\
x+3y+6z+2t &= 3,\\
3x+2y+2z+2t &= 1,\\
2x+y+2z &= 4,\\
4x+5y+z+4t &= 4,\\
5x+5y+3z+2t &= 4.
\end{aligned}$$

3. Solve the system of linear equations (a is a real parameter)

a)

$$\begin{aligned} 4x+\ y+\ 2z &= 0,\\ x+ay-\ \ z &= 0,\\ 6x+\ y+2az &= 0. \end{aligned}$$

b)

$$\begin{aligned} x+3y+\ z &= 1,\\ 2x-\ y-3z &= 0,\\ 3x+ay-2z &= 3. \end{aligned}$$

c)

$$\begin{aligned} 2x+\ y+\ z &= 6-a,\\ 2x+3y+2z &= 11+5a,\\ 2x+2y+3z &= 7+8a. \end{aligned}$$

d)

$$\begin{aligned} x+\ y+az+\ u &= a,\\ ax+\ y+\ z+\ u &= a,\\ x+\ y+\ z+au &= 1,\\ x+ay+\ z+\ u &= 1. \end{aligned}$$

e)

$$\begin{aligned} 5x-3y+2z+\ 4u &= 3,\\ 4x-2y+3z+\ 7u &= 1,\\ 8x-6y-\ z-\ 5u &= 9,\\ 7x-3y+7z+17u &= a. \end{aligned}$$

f)

$$\begin{aligned} ax+\ y+\ z+u &= 1,\\ x+ay+\ z+u &= a,\\ x+\ y+az+u &= a^2. \end{aligned}$$

g)

$$\begin{aligned} 2x+3y-\ z &= 0,\\ ax+4y+2z &= 0. \end{aligned}$$

h)

$$\begin{aligned} ax- \; 2y+ \qquad\quad z &= 0, \\ 3x+2ay- \qquad\quad z &= 0, \\ a^2x+ \quad y+(a-1)z &= 0. \end{aligned}$$

i)

$$\begin{aligned} (a+1)x+ \qquad y+ \qquad z &= a^2+3a, \\ x+(a+1)y+ \qquad z &= a^3+3a^2, \\ x+ \qquad y+(a+1)z &= a^4+3a^3. \end{aligned}$$

4. Solve the system of linear equations (a, b are real parameters)

a)

$$\begin{aligned} ax+ \; by+ \; z &= 1, \\ x+aby+ \; z &= b, \\ x+ \; by+az &= 1. \end{aligned}$$

b)

$$\begin{aligned} ax+ \qquad y+ \qquad z &= 1, \\ (b-1)by+ \qquad z &= 0, \\ 2ax+ \qquad 2by+(b+5)z &= 2b. \end{aligned}$$

5. Solve the system of linear equations (a, b, c are real parameters)

a)

$$\begin{aligned} x+ \quad y+ \quad z &= 1, \\ ax+ \; ay+ \; bz &= c, \\ a^2x+a^2y+b^2z &= c^2. \end{aligned}$$

b)

$$\begin{aligned} ax+ \; y+ \; z &= 1, \\ x+by+ \; z &= 1, \\ x+ \; y+cz &= 2. \end{aligned}$$

6. Solve the system of linear equations (a, b, c, d are real parameters)

a)

$$\begin{aligned} x+y+ &= a, \\ y+z+ &= b, \\ z+u &= c, \\ x+v &= d. \end{aligned}$$

b)

$$\begin{aligned} ax+\ y+\ z &= b, \\ x+ay+\ z &= c, \\ x+\ y+az &= d. \end{aligned}$$