Exponential model

The model describes a continuous nonnegative variable. Its pdf is

$$f\left(y|a\right)=a\exp\left(-ay\right),\ a>0$$

Posterior pdf

$$f\left(a|y\left(t\right)\right)=a^{\kappa_t}\exp\left(-aS_t\right)$$

with statistics update

$$S_t=S_{t-1}+y_t$$
$$\kappa_t=\kappa_{t-1}+1$$

Point estimate of a

$$\hat{a}_t=\frac{\kappa_t}{S_t}=\frac{1}{\bar{y}}$$

Program and its description

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Derivations

Product of models

$$f\left(y_1|a\right) f\left(y_2|a\right) = a\exp\left(-ay_1\right) a\exp\left(-ay_2\right) =$$

$$= a^2\exp\left(-a\left(y_1 + y_2\right)\right)$$

from which

$$\prod_{i=1}^{n} f\left(y_i|a\right) = a^n \exp\left(-a\sum_{i=1}^{n} y_i\right)$$

where $S_n = \sum_{i=1}^{n} y_i$ a $\kappa_t = n$

Posterior pdf

$$f\left(a|y\left(t\right)\right)=a^{\kappa_t}\exp\left(-aS_t\right)$$

with update

$$S_t = S_{t-1}+y_t$$
$$\kappa_t = \kappa_{t-1}+1$$

starting with $S_0=0$, $\kappa_0=0$.

Bayes $f\left(a|y\left(t\right)\right)\propto f\left(y_t|a\right)f\left(a|y\left(t-1\right)\right)$

$$a^t\exp\left(-a\sum_{i=1}^{t}y_i\right)=a\exp\left(-ay_t\right)\,a^{t-1}\exp\left(-a\sum_{i=1}^{t-1}y_i\right)$$

and formally

$$a^{\kappa_t}\exp\left(-aS_t\right)=a\exp\left(-ay_t\right)\,a^{\kappa_{t-1}}\exp\left(-aS_{t-1}\right)$$

Point estimate (ML)

$$\frac{d}{da} a^{\kappa} \exp\left(-aS\right) = \kappa a^{\kappa-1} \exp\left(-aS\right) - a^{\kappa} S \exp\left(-aS\right) = 0$$

$$\kappa - aS = 0 \quad \rightarrow \quad a = \frac{\kappa}{S} = \frac{1}{\bar{y}}$$

Prior information

We choose

$$f\left(a|y\left(0\right)\right)=a^{\kappa_0}\exp\left(-aS_0\right)$$

where S_0 is sum of prior data and κ_0 is their count.

If we have information that $a \doteq 5$, i.e. $y=\frac{1}{5}$, first we chose the strength, e.g. $\kappa_0=20$ and then $S_0=\sum y=\kappa_0\frac{1}{5}=4$.

From it $\frac{1}{a}=\frac{S_0}{\kappa_0}=\frac{4}{20}=\frac{1}{5}$ and

Shifted exponential model

It is an extension of the exponential model for data which are greater then some value $b \in R$. In other words, it is a shifted exponential model by the value b, where b is another estimated parameter of the model which has the form

$$f\left(y|a,b\right) = a\exp\left\{-a\left(y-b\right)\right\}$$

where $y \geq 0$, $a > 0$, $b \in R$.

Model

$$y_t = a\exp\left\{-a\left(y_t-b\right)\right\},\ y_t \geq b,\ a > 0$$

Expectation

$$E\left[y\right] = b + \frac{1}{a}$$

Variance

$$D\left[y\right] = \frac{1}{a^2}$$

Method of moments

Expectation and variance

$$E\left[y\right] = \bar{y},\quad D\left[y\right] = s_y^2 = \bar{y^2} - \bar{y}^2$$

Standard deviation

$$s_y = \sqrt{\bar{y^2} - \bar{y}^2}$$

from which we get the estimates

$$\hat{a} = \frac{1}{s_y}, \text{ and } \hat{b} = \bar{y} - s_y$$

Algorithm of recursive estimation

s1=0; //sum of y

s2=0; // sum of y^2

for $t = 1 : nd$

s1=s1+y_t

s2=s2+y_t^2

s=sqrt(s2/t-(s1/t)^2

a=1/s

b=(s1/t)-s

end

Program

```
// expShift1MM.sce
// Odhad parametrů posunuté exponenciály
```

```
//            y = a.exp(-a(x-b))
// -------------------------------------
clc, clear, close, mode(0)
exec SCIHOME/ScIntro.sce

nd=200;

a=.8;
b=3;

// simulation
for t=1:nd
  y(t)=-log(rand(1,1,'u'))/a+b;
end

// estimation
s1=0;
s2=0;
for t=1:nd
  s1=s1+y(t);
  s2=s2+y(t)**2;
  sd=sqrt(s2/t-(s1/t)**2);
  a(t)=1/sd;
```

```
  b(t)=s1/t-sd;
end

m=mean(y);
sd=stdev(y);

plot([a b])
```

Maximum likelihood method

Likelihood

$$L = a^{\kappa} \exp\left\{-a\left(S-b\right)\right\}$$

where $S = \sum y$ and κ is counter

$$S_t = S_{t-1} + y_t$$
$$\kappa_t = \kappa_{t-1} + 1$$

Maximum

$$\frac{dL}{db} = a^{\kappa+1} \exp\left\{-a\left(S-b\right)\right\} \;\rightarrow\; \text{nothing}$$
$$\frac{dL}{da} = \kappa a^{\kappa-1} \exp\left\{-a\left(S-b\right)\right\} - a^{\kappa}\left(S-b\right) \exp\left\{-a\left(S-b\right)\right\} \;\rightarrow$$

$$\hat{a} = \frac{1}{\bar{y} - b}, \text{ where } \bar{y} = \frac{S}{\kappa}$$

Conclusion

Estimation of b is not possible, the estimate of a needs $\hat{b}$. So ML fails.

A possibility is, to estimate a for $b = 0$ and to estimate b similarly as for the uniform model - as the minimum of data.

Then

$$d_y = \hat{b}_{t-1} - y_t$$

$$\hat{b}_t = b - d_y$$

and

$$\hat{a}_t = \frac{1}{\bar{y}_t - \hat{b}_t}$$

where $\bar{y}_t = \sum_{\tau=1}^{t} y_\tau$ is the average of data up to now.

Program

```
// expShift1ML.sce
// Odhad parametrů posunuté exponenciály
//           y = a.exp(-a(x-b))
// -------------------------------------
clc, clear, close, mode(0)
exec SCIHOME/ScIntro.sce
```

```
nd=2000;

a=.8;
b=3;

// simulation
for t=1:nd
  y(t)=-log(rand(1,1,'u'))/a+b;
end

// estimation
s1=0;
s2=0;
for t=1:nd
  s1=s1+y(t);
  s2=s2+y(t)**2;
  sd=sqrt(s2/t-(s1/t)**2);
  b(t)=s1/t-sd;
  a(t)=1/(s1/t-b(t));
end

m=mean(y);
```

```
sd=stdev(y);

set(scf(),'position',[600 100 500 400])
plot([a b])

Ea=a($), Eb=b($)
```