Linear Programming

B

Before studying this supplement you should know or, if necessary, review

1. Competitive priorities, Chapter 2
2. Capacity management concepts, Chapter 9
3. Aggregate planning, Chapter 13
4. Developing a master schedule, Chapter 14

LEARNING OBJECTIVES

After studying this supplement, you should be able to

1. Describe the role of mathematical models in operations decision making.
2. Describe constrained optimization models.
3. Understand the advantages and disadvantages of using optimization models.
4. Describe the assumptions of linear programming.
5. Formulate linear programs.
6. Describe the geometry of linear programs.
7. Describe the graphical solution approach.
8. Use the simplex algorithm.
9. Use artificial variables.
10. Describe computer solutions of linear programs.
11. Use linear programming models for decision making.

SUPPLEMENT OUTLINE

■ THE ROLE OF MATHEMATICAL MODELS IN OPERATIONS DECISION MAKING

► A **model** represents the essential features of an object, system, or problem without unimportant details.

Advances in business and engineering research and computer technology have expanded managers' use of mathematical models. A **model** represents the essential features of an object, system, or problem without unimportant details. The models in this supplement have the important aspects represented in mathematical form using variables, parameters, and functions. Analyzing and manipulating the model gives insight into how the real system behaves under various conditions. From this we determine the best system design or action to take.

Mathematical models are cheaper, faster, and safer than constructing and manipulating real systems. Suppose we want to find the mixture of recycled scrap paper to use when producing a type of paperboard that minimizes cost. A company could try several different combinations, check the quality, and calculate the cost. Since all possible combinations are not tried, the optimum combination will probably not be found. Alternatively, using a mathematical model, we evaluate all possible combinations to find the one that satisfies product specifications at the lowest price. Mathematical modeling is quicker and less expensive than using the trial-and-error approach.

► **Constrained optimization models**
Math models that find the best solution with respect to some evaluation criterion.

Facility location, vehicle routing and scheduling, personnel, machine and job scheduling, product mixes, and inventory management problems are formulated as constrained optimization models. **Constrained optimization models** are mathematical models that find the best solution with respect to some evaluation criterion from a set of alternative solutions. These solutions are defined by a set of mathematical constraints—mathematical inequalities or equalities.

■ CONSTRAINED OPTIMIZATION MODELS

Constrained optimization models have three major components: decision variables, objective function, and constraints.

► **Decision variables**
Physical quantities controlled by the decision maker.

1. *Decision variables* are physical quantities controlled by the decision maker and represented by mathematical symbols. For example, the decision variable x_j can represent the number of pounds of product j that a company will produce during some month. Decision variables take on any of a set of possible values.

► **Objective function**
Evaluation criterion.

2. *Objective function* defines the criterion for evaluating the solution. It is a mathematical function of the decision variables that converts a solution into a numerical evaluation of that solution. For example, the objective function may measure the profit or cost that occurs as a function of the amounts of various products produced. The objective function also specifies a direction of optimization, either to maximize or minimize. An optimal solution for the model is the best solution as measured by that criterion.

► **Constraints**
Physical, economic, technological, legal, ethical, or other limits on what numerical values can be assigned to the decision variables.

3. *Constraints* are a set of functional equalities or inequalities that represent physical, economic, technological, legal, ethical, or other restrictions on what numerical values can be assigned to the decision variables. For example, constraints might ensure that no more input is used than is available. Constraints can be definitional, defining the number of employees at the start of a period $t + 1$ as equal to the number of employees at the start of period t, plus those added during period t minus those leaving the organization during period t. In constrained optimization models we find values for the

decision variables that maximize or minimize the objective function and satisfy all constraints.

The following example shows how an operational problem can be represented and analyzed using a constrained optimization model.

■ Example B.1 The Healthy Pet Food Company Product Mix

The Healthy Pet Food Company manufactures two types of dog food: Meaties and Yummies. Each package of Meaties contains 2 pounds of cereal and 3 pounds of meat; each package of Yummies contains 3 pounds of cereal and 1.5 pounds of meat. Healthy believes it can sell as much of each dog food as it can make. Meaties sell for $2.80 per package and Yummies sell for $2.00 per package. Healthy's production is limited in several ways. First, Healthy can buy only up to 400,000 pounds of cereal each month at $0.20 per pound. It can buy only up to 300,000 pounds of meat per month at $0.50 per pound. In addition, a special piece of machinery is required to make Meaties, and this machine has a capacity of 90,000 packages per month. The variable cost of blending and packing the dog food is $0.25 per package for Meaties and $0.20 per package for Yummies. This information is given in Table B-1.

Table B-1   Healthy Pet Food Data

	Meaties	Yummies
Sales price per package	$2.80	$2.00
Raw materials per package		
Cereal	2.0 lb.	3.0 lb.
Meat	3.0 lb.	1.5 lb.
Variable cost—blending and packing	$0.25 package	$0.20 package
Resources		
Production capacity for Meaties	90,000 packages per month	
Cereal available per month	400,000 lb.	
Meat available per month	300,000 lb.	

Suppose you are the manager of the Dog Food Division of the Healthy Pet Food Company. Your salary is based on division profit, so you try to maximize its profit. How should you operate the division to maximize its profit and your salary?

Solution:
The Decision Variables. We first identify those things over which we have control: the *decision variables.* In this problem we have direct control over two quantities: the number of packages of Meaties to make each month, and the number of packages of Yummies to make each month. Within the model these two quantities appear repeatedly, so we represent them in a simple fashion. We designate these variables by the symbols M and Y.

M = number of packages of Meaties to make each month
Y = number of packages of Yummies to make each month

Note that the amount of meat used each month and the amount of cereal used each month are not good choices for the variables. First, we control these only indirectly through our choice of M and Y. More important, using these as variables could lead to ambiguous production plans. Determining how much cereal and meat to use in production does not tell us how to use it—how much of each dog food to make. In contrast, after determining the values for M and Y, we know what to produce and how much meat and cereal are needed.

Objective function. Any pair of numerical values for the variables M and Y is a production plan. For example, $M = 10{,}000$ and $Y = 20{,}000$ means we make 10,000 packages of Meaties and 20,000 packages of Yummies each month. But how do we know whether this is a good production plan? We need to specify a criterion for evaluation—an *objective function.* The most appropriate objective function is to maximize monthly profit. (Actually, this is the contribution to profit: fixed costs are ignored because any plan that maximizes revenue minus variable costs maximizes profit as well.) The profit earned by Healthy is a direct function of the amount of each dog food made and sold, the decision variables. Monthly profit, designated as z, is written as follows:

$z = $ (profit per package of Meaties) $\times$ (number of packages of Meaties made and sold monthly) $+$ (profit per package of Yummies) $\times$ (number of packages of Yummies made and sold monthly)

The profit per package for each dog food is computed as follows:

	Meaties	**Yummies**
Selling price	2.80	2.00
Minus		
Meat	1.50	0.75
Cereal	0.40	0.60
Blending	0.25	0.20
Profit per package	0.65	0.45

We write the month profit as

$z = 0.65M + 0.45Y$

Constraints. If we want to make z as large as possible, why not make M and Y equal to infinity and earn an infinite profit? We cannot do this because there are limits on the availability of cereal and meat and on the production capacity for Meaties. (In reality, there is also a limit on demand, but we ignore that here for simplicity.) We want to maximize z, but subject to satisfying the stated constraints. To solve the problem, we express these constraints as mathematical equalities or inequalities. Let's begin with the availability of cereal constraint:

(The number of lb. of cereal used in production each month) $\leq 400{,}000$ lb.

The left-hand side (l.h.s.) of the constraint is determined by the number of packages of Meaties and Yummies made. Specifically, the l.h.s. is

(lb. of cereal per package of Meaties) $\times$ (packages of Meaties made and sold monthly) $+$ (lb. of cereal per package of Yummies) $\times$ (packages of Yummies made and sold monthly)

Substituting the cereal content for each product and the decision variables into this expression, we write the constraint as

$2M + 3Y \leq 400{,}000$

Using similar reasoning, the restriction on the availability of meat is expressed as

$3M + 1.5Y \leq 300{,}000$

In addition to these constraints, the number of packages of Meaties produced each month can not exceed 90,000; that is,

$M \leq 90{,}000$

Finally, negative production levels do not make sense, so we require that $M \geq 0$ and $Y \geq 0$. Putting all these together gives the following constrained optimization model.

Maximize $z = 0.65M + 0.45Y$

Subject to
$$
\begin{aligned}
2M + 3Y &\leq 400{,}000 \\
3M + 1.5Y &\leq 300{,}000 \\
M &\leq 90{,}000 \\
M, Y &\geq 0
\end{aligned}
$$

This type of model is called a linear programming model or a **linear program** because the objective function is linear and functions in all the constraints are linear. The optimum solution for the Healthy Pet Food problem is $M = 50{,}000$, $Y = 100{,}000$, and $z = \$77{,}500$. That is, Healthy should make 50,000 packages of Meaties and 100,000 packages of Yummies each month, and it will earn a monthly profit of $77,500. Before discussing linear programming in detail, let's consider the advantages and disadvantages of optimization models in general.

▶ A **linear program** has a linear objective function and linear constraints.

ADVANTAGES AND DISADVANTAGES OF USING OPTIMIZATION MODELS

The main benefit of optimization models is the ability to evaluate possible solutions in a quick, safe, and inexpensive way without actually constructing and experimenting with them. Other benefits are as follows.

1. *Structures the thought process*. Constructing an optimization model of a problem forces a decision maker to think through the problem in a concise, organized fashion. The decision maker determines what factors he or she controls; that is, what the decision variables are. The decision maker specifies how the solution will be evaluated (the objective function). Finally, the decision maker describes the decision environment (the constraints). Modeling acts as a way of organizing and clarifying the problem.
2. *Increases objectivity*. Mathematical models are more objective since all assumptions and criteria are clearly specified. Although models reflect the experiences and biases of those who construct them, these biases can be identified by outside observers. By using a model as a point of reference, the parties can focus their discussion and disagreements on its assumptions and components. Once the model is agreed on, people tend to live by the results.
3. *Makes complex problems more tractable*. Many problems in managing an organization are large and complex and deal with subtle, but significant, interrelationships among organizational units. For example, in determining the optimal amounts of various products to ship from geographically dispersed warehouses to geographically dispersed customers and the routes that should be taken, the human mind cannot make the billions of simultaneous trade-offs that are necessary. In these cases, the decision maker often uses simple rules of thumb, which can result in less than optimal solutions. Optimization models make it easier to solve complex organization-wide problems.
4. *Make problems amenable to mathematical and computer solution*. By representing a real problem as a mathematical model, we use mathematical solution and analysis techniques and computers in a way that is not otherwise possible.

5. ***Facilitates "what if" analysis***. Mathematical models make it relatively easy to find the optimal solution for a specific model and scenario. They also make "what if" analysis easy. With "what if" analysis, we recognize that the prices, demands, and product availabilities assumed in constructing the model are simply estimates and may differ in practice. Therefore, we want to know how the optimal solution changes as the value of these parameters vary from the original estimates. That is, we want to know how sensitive the optimal solution is to the assumptions of the model. "What if" analysis is also called ***sensitivity*** or ***parametric analysis***.

▶ **Sensitivity analysis** allows the decision maker to perform "what if?" analysis.

Although mathematical modeling has many advantages, there are also disadvantages. The actual formulation or construction of the model is the most crucial step in mathematical modeling. Since the problems tend to be very complex, it is possible to mismodel the real problem. Important decision variables or relationships may be omitted or the model may be inappropriate for the situation. *The optimal solution to the wrong problem is of no value.*

A second disadvantage is not understanding the role of modeling in the decision-making process. The optimal solution for a model is not necessarily the optimal solution for the real problem. Mathematical models are tools to help us make good decisions. However, they are not the only factor that should go into the final decision. Sometimes the model only evaluates solutions with regard to quantitative criteria. In these cases qualitative factors must also be considered when making the final decision.

The bottom line for evaluating a model is whether or not it helps a decision maker identify and implement better solutions. The model should increase the decision maker's confidence in the decision and the willingness to implement it.

■ ASSUMPTIONS OF LINEAR PROGRAMMING MODELS

Linear programs are constrained optimization models that satisfy three requirements.

1. The decision variables must be continuous; they can take on any value within some restricted range.
2. The objective function must be a linear function.
3. The left-hand sides of the constraints must be linear functions.

Thus, linear programs are written in the following form:

Maximize or minimize $\quad z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \begin{array}{c} \leq \\ = \\ \geq \end{array} b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \begin{array}{c} \leq \\ = \\ \geq \end{array} b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \begin{array}{c} \leq \\ = \\ \geq \end{array} b_m$$

▶ **Parameters**
Constants given in the problem assumptions.

where the x_j values are decision variables and c_j, a_{ij}, and b_i values are constants, called **parameters** or **coefficients**, that are given or specified by the problem assumptions. Most linear programs require that all decision variables be nonnegative.

Linear programs make the following implicit assumptions.

1. *Proportionality*. With linear programs, we assume that the contribution of individual variables in the objective function and constraints is proportional to their value. That is, if we double the value of a variable, we double the contribution of that variable to the objective function and each constraint in which the variable appears. The contribution per unit of the variable is constant. For example, suppose the variable x_j is the number of units of product j produced and c_j is the cost per unit to produce product j. If doubling the amount of product j produced doubles its cost, the per unit cost is constant and the proportionality assumption is satisfied.

2. *Additivity*. Additivity means that the total value of the objective function and each constraint function is obtained by adding up the individual contributions from each variable.

3. *Divisibility*. The decision variables are allowed to take on any real numerical values within some range specified by the constraints. That is, the variables are not restricted to integer values. When fractional values do not make a sensible solution, such as the number of flights an airline should have each day between two cities, the problem should be formulated and solved as an integer program.

4. *Certainty*. We assume that the parameter values in the model are known with certainty or are at least treated that way. The optimal solution obtained is optimal for the specific problem formulated. If the parameter values are wrong, then the resulting solution is of little value.

In practice, the assumptions of proportionality and additivity need the greatest care and are most likely to be violated by the modeler. With experience, we recognize when integer solutions are needed and the variables must be modeled explicitly.

▶ **Proportionality** means that the contribution of individual variables in the objective function is proportional to their value.

▶ **Additivity** means the total value of the objective function and each constraint is the sum of the individual contributions from each variable.

▶ **Divisibility** means the decision variables can take on any real numerical values within a specified range.

▶ **Certainty** means the parameters are known.

FORMULATING LINEAR PROGRAMS ◼

This section presents simple examples of real managerial problems that can be formulated as linear programs. Each example has a name describing the type of problem. In real life, problems are seldom as pure and clean as these examples. Do not try to memorize and match the problems illustrated here with real problems you may encounter. In practice, problems may contain a mixture of features from several of the categories illustrated here. You should focus on why and how various physical relationships are best represented in model form.

Model formulation is the most important and the most difficult aspect of solving a real problem. Solving a model that does not accurately represent the real problem is useless. There is no simple way to formulate optimization problems, but the following suggestions may help.

Steps in Problem Formulation

1. *Identify and define the decision variables for the problem*. Define the variables completely and precisely. All units of measure need to be stated explicitly, including time units if appropriate. For example, if the variables represent quantities of a product produced, these should be defined in terms of tons per hour, units per day, barrels per month, or some other appropriate units.

2. ***Define the objective function***. Determine the criterion for evaluating alternative solutions. The objective function will normally be the sum of terms made up of a variable multiplied by some appropriate coefficient (parameter). For example, the coefficients might be profit per unit of production, distance travel per unit transported, or cost per person hired.

3. ***Identify and express mathematically all of the relevant constraints***. It is often easier to express each constraint in words before putting it into mathematical form. The written constraint is decomposed into its fundamental components. Then substitute the appropriate numerical coefficients and variable names for the written terms. A common mistake is using variables that have not been defined in the problem, which is not valid. This mistake is frequently caused by not defining the original variables precisely. The formulation process is iterative, and sometimes additional variables must be defined or existing variables redefined. For example, if one of the variables is the total production of the company and five other variables represent the production at the company's five plants, then there must be a constant that forces total production to equal the sum of the production at the plants.

Let's look at the formulation process for typical operations problems.

Feed Mix or Diet Problem

One of the first problems solved using linear programming is the feed mix problem, which is illustrated in Example B.2.

■ Example B.2 International Wool Company Feed Mix Problem

International Wool Company operates a large farm on which sheep are raised. The farm manager determined that for the sheep to grow in the desired fashion, they need at least minimum amounts of four nutrients (the nutrients are nontoxic so the sheep can consume more than the minimum without harm). The manager is considering three different grains to feed the sheep. Table B-2 lists the number of units of each nutrient in each pound of grain, the minimum daily requirements of each nutrient for each sheep, and the cost of each grain. The manager believes that as long as a sheep receives the minimum daily amount of each nutrient, it will be healthy and produce a standard amount of wool. The manager wants to raise the sheep at minimum cost.

Table B-2   International Wool Data

			Grain		Minimum Daily Requirement (units)
		1	2	3	
Nutrient	A	20	30	70	110
Nutrient	B	10	10	0	18
Nutrient	C	50	30	0	90
Nutrient	D	6	2.5	10	14
Cost (¢/lb)		41	36	96	

Solution

The quantities that the manager controls are the amounts of each grain to feed each sheep daily. We define

x_j = number of pounds of grain j (= 1, 2, 3) to feed each sheep daily

Note that the units of measure are completely specified. In addition, the variables are expressed on a per sheep basis. If we minimize the cost per sheep, we minimize the cost for any group of sheep. The daily feed cost per sheep will be

(cost per lb of grain j) × (lb. of grain j fed to each sheep daily)

That is, the objective function is to

Minimize $z = 41x_1 + 36x_2 + 96x_3$

Why can't the manager simply make all the variables equal to zero? This keeps costs at zero, but the manager would have a flock of dead sheep, because there are minimum nutrient constraints that must be satisfied. The values of the variables must be chosen so that the number of units of nutrient A consumed daily by each sheep is equal to or greater than 110. Expressing this in terms of the variables yields

$20x_1 + 30x_2 + 70x_3 \geq 110$

The constraints for the other nutrients are

$$10x_1 + \quad\quad 10x_2 \geq 18$$
$$50x_1 + \quad\quad 30x_2 \geq 90$$
$$6x_1 + 2.5x_2 + 10x_3 \geq 110$$

and finally

all x_js ≥ 0

The optimal solution to this problem (obtained using a computer software package) is $x_1 = 0.595, x_2 = 2.008, x_3 = 0.541,$ and $z = 148.6$ cents.

It is common practice to take a model initially used for one application and apply it to other situations. The feed mix problem is a good example of a case where one might use the same basic structure of a model in different applications. For example, a golf course manager can use the model to select the best mix of fertilizers to provide the grass with the desired amounts of active chemicals (nitrogen, phosphorus, potash). The manager's problem is structurally the same as that faced by the manager of International Wool.

Although the basic structure of one model may be appropriate for another application, frequently the model needs modification. For example, suppose the U.S. Army decides to use the feed mix model to select a cost-minimizing diet for its soldiers that satisfies minimum nutritional requirements. The basic feed mix problem makes several subtle assumptions that do not apply for humans. First, issues of taste have been ignored. Earlier, we assumed that the sheep will eat whatever grain mixture we feed them. Humans have varying tastes to consider. Some foods may not taste good together. Second, not all soldiers are of similar size or have the same appetite. Third, the basic feed mix is a static model: the optimal feed mix today is the same as that of tomorrow and the next 500 days unless some parameters change. We do not want to feed people the same meal day after day. Let's look at another type of problem.

Blending Problem

In the Healthy Pet Food example, a product mix problem, the company determines how much of various products to make. The mixture of inputs used in each product is fixed. In many industries such as the oil, chemical, paper, metals, and food processing industries, a company controls how much of a product to make and the mix of inputs to use in making it. This is called a blending problem.

■ **Example B.3 Solar Oil Company Blending Problem**

Solar Oil Company is a gasoline refiner and wholesaler. It sells two products to gas stations: regular and premium gasoline. It makes these two final products by blending together four raw gasolines and some chemical additives (the amount and cost of the additives per barrel are assumed to be independent of the mixture). Each gasoline has an octane rating that reflects its energy content. Table B-3 lists the octane, purchase price per barrel, and availability at that price per day. This table also gives the required minimum octane for each final gasoline, the net selling price per barrel (removing the cost of the additives), and the expected daily demand for gas at that price. Solar Oil can sell all the gas it produces up to that amount.

The blending of gasoline is approximately a linear operation in terms of volume and octane. If x barrels of 80 octane gasoline are blended with y barrels of 90 octane gasoline, this produces $x + y$ barrels of gasoline with an octane of $(80x + 90y)/(x + y)$. There is no significant volume gain or loss, and octane of the mixture is a weighted average of the octanes of the inputs.

Table B-3   Solar Oil Data

		Octane	Cost ($/b)	Available daily
Raw	1	86	17.00	20,000
gasolines	2	88	18.00	15,000
	3	92	20.50	15,000
	4	96	23.00	10,000
		Octane	Price ($/b)	Maximum daily demand
Products	Regular	89	19.50	35,000
	Premium	93	22.00	23,000

Solution:
The manager of Solar Oil's operation wants to maximize the company's profit. The first question is: What quantities does the manager control? What can the manager manipulate to influence profit? It is incomplete simply to say that the manager controls the amount of each final product to make. The manager controls, and must determine, how to make each final product and how much to make. This can be expressed by letting x_{ij} = number of barrels of raw gas I ($= 1, 2, 3, 4$) used per day to make final product j ($= R, P$) be the decision variables. Each barrel of raw gas i that is blended in final product j and then sold generates a profit equal to its selling price minus its cost. The objective function is the sum of all terms of the form

(profit per barrel of raw gas i that is blended into gas j) × (number of barrels of raw gas i blended into gas j per day)

Substituting for these gives

Maximize $z = 2.5x_{1R} + 1.5x_{2R} - x_{3R} - 3.5x_{4R} + 5.0x_{1P} + 4.0x_{2P} + 1.5x_{3P} - x_{4P}$

Note that the coefficients for some variables are negative. For example, Solar loses $1.00 on each barrel of raw gas 4 that is blended into premium. Does this imply that the optimal value for these variables must be zero and that they can be dropped from the problem? No! In blending operations, it is common for some low-cost materials to be combined with high-cost materials. Although it appears that we are losing money on the high-cost materials, they make the low-cost materials more valuable, and often the final product cannot be made without them. For example, tungsten steel combines low-cost iron ore or scrap (worth $100/ton) with tungsten (costing thousands of dollars per ton) to make steel that might sell for $500 per ton. The manufacturer loses money on the tungsten (on a per ton basis) but is more than compensated by the enhanced value of the iron ore. Therefore, we do not omit variables from the problem unless we can prove that their optimum value is zero.

The next step is to identify the constraints. The availability constraint for each raw gasoline is

barrels of raw gas i used per day $\leq$ barrels of gas i available per day

The number of barrels of raw gas i used each day is the amount used to make regular gasoline x_{iR} plus the amount used each day to make premium gasoline x_{iP}. The availability constraints can be written as

$$x_{1R} + x_{1P} \leq 20{,}000$$
$$x_{2R} + x_{2P} \leq 15{,}000$$
$$x_{3R} + x_{3P} \leq 15{,}000$$
$$x_{4R} + x_{4P} \leq 10{,}000$$

The demand constraints put an upper limit on how much regular and premium gasoline can be sold. The total amount made of each is the sum of the raw gasolines allocated to making each gasoline each day. In other words,

$$x_{1R} + x_{2R} + x_{3R} + x_{4R} \leq 35{,}000$$
$$x_{1P} + x_{2P} + x_{3P} + x_{4P} \leq 23{,}000$$

If the model formulation is left at this stage, the optimal solution is to mix the lowest-cost gasolines into the final products, regardless of octane. Therefore, we need to include constraints that guarantee the variables will take on values that produce final gasolines with at least the minimum specified octane ratings. The octane rating of the regular gasoline that is produced will be a weighted average of the octanes of the raw gasolines used; that is,

octane of regular = [86 (barrels of raw gas 1 used/day to make regular) + 88 (barrels of raw gas 2 used/day to make regular) + · · · + 96 (barrels of raw gas 4 used/day to make regular)]/[total barrels of raw gases blended into regular gasoline]

which should be at least 89. Substituting the appropriate variable names for these quantities produces the constraint

$$[86x_{1R} + 88x_{2R} + 92x_{3R} + 96x_{4R}]/[x_{1R} + x_{2R} + x_{3R} + x_{4R}] \geq 89$$

Multiplying both sides by $(x_{1R} + x_{2R} + x_{3R} + x_{4R})$ and then gathering terms so that variables appear only once and all are on the left-hand side yields

$$-3x_{1R} - x_{2R} + x_{3R} + 7x_{4R} \geq 0$$

Using the same approach to guarantee an octane of 93 for premium gas produces the constraint

$$-7x_{1R} - 5x_{2R} - x_{3R} + 3x_{4R} \geq 0$$

Finally, all variables should be nonnegative in value. The optimal solution to this linear program is $x_{1R} = 11{,}428.57$, $x_{2R} = 9107.14$, $x_{3R} = 14{,}464.29$, $x_{4R} = 0$, $x_{1P} = 0$, $x_{2P} = 5892.86$, $x_{3P} = 535.71$, $x_{4P} = 10{,}000$, and $z = \$42{,}142.86$ daily.

In this example, fractional amounts of raw gasoline input are perfectly reasonable. Fluids can be finely measured and the solution represents a flow rate. However, even for a simple problem of this sort, the optimal solution is far from obvious. With linear programming the daily profit might be 3 to 5% better than that achieved using an intelligent seat-of-the-pants approach. A 3% savings represents approximately $350,000 to $400,000 per year. Not bad, for using a simple model. As raw material prices change, the optimal blend and output levels are quickly revised.

Multiperiod Planning Problem

Linear programming is used extensively for planning and scheduling of operations. One form of planning is called aggregate planning, which concentrates on scheduling production, personnel, and inventory levels during intermediate-term planning horizons. The following example is a simple version of aggregate planning.

■ Example B.4 Basel Tool and Die Company Multiperiod Planning Problem

Basel Tool and Die Company (BTD) makes large industrial pipe wrenches in one of its factories. The marketing department estimates demand for this product during the next 6 months to be:

Month	Demand
January	370
February	430
March	380
April	450
May	520
June	440

With the current labor force, BTD believes it can make approximately 420 pipe wrenches per month at a cost of $40 per wrench using regular-time production. An additional 80 wrenches per month can be made using overtime production at a cost per wrench of $45. Wrenches can be made in advance and held in inventory for later shipment at a cost of $3 per month per wrench. The monthly demand for wrenches must be satisfied every month. At the end of December (beginning of January) BTD has 10 wrenches in inventory. BTD wants to plan its production, including overtime, and inventory for the next 6 months so as to maximize profit. Assuming the revenue for these wrenches is fixed, the production manager can maximize profit by minimizing the total costs incurred in producing and delivering the wrenches.

Solution:
The quantities that the decision maker controls are (1) the number of wrenches to make each month using regular-time production, (2) the number of wrenches to make each month using overtime production, and indirectly (3) the number of wrenches to keep in inventory at the end of each month. We define our decision variables as follows (to keep a clear time convention, we assume that wrenches are made during a month; at the end of the month, wrenches are shipped to customers; any wrench not shipped incurs a holding cost for that month):

R_t = number of wrenches made during month t using regular-time production
O_t = number of wrenches made during month t using overtime production
I_t = number of wrenches in inventory at the end of month t

where for each variable, $t = 1, \cdots, 6$. Note that for wrenches kept in inventory there is no need to keep track of when they were made.

The objective is to choose the values of the variables to minimize the total cost incurred during the next 6 months. (If the time value of money is more important in the problem, the present value of the costs for each month are used.) The total cost is made up of the production cost of making the wrenches, during both regular time and overtime, and the inventory cost. The objective can then be written as follows:

Minimize $$z = \sum_{t=1}^{6} (40\,R_t + 45\,O_t + 3\,I_t)$$

The easiest constraints to represent are those limiting the amount of regular-time and over-time production each month:

$R_t \le 420$ for $t = 1, \cdots, 6$

and

$O_t \le 80$ for $t = 1, \cdots, 6$

In addition, we need constraints guaranteeing that demand is satisfied each month. At this point the model becomes tricky. We might think it would be sufficient adding a constraint for each month stating that beginning inventory for that month plus total production must be at least as large as the demand for that month; for example,

$10 + R_1 + O_1 \ge 370$
$I_1 + R_2 + O_2 \ge 430$

and so forth. However, if we solve the problem as it stands, the optimal solution would be to let $R_1 = 360$, and all other R_t values and all O_t values equal zero, $I_1 = 430$, $I_2 = 380$, $I_3 = 450$, $I_4 = 520$, $I_5 = 440$, and $I_6 = 0$. This means we should satisfy the demands in the last 5 months using inventories (because inventories are cheap but producing wrenches is expensive). However, this makes no sense because inventories are the result of excess production. This problem occurs because there are no constraints defining what the relationship is among inventories, production, and demand. Therefore, we need constraints that define inventories as follows:

(Wrenches in inventory at beginning of month t) + (wrenches made during month t) = (wrenches shipped at end of month t) + (wrenches in inventory at end of month t)

Converting these to mathematical form, we get the constraints

$10 + R_1 + O_1 = 370 + I_1$		$R_1 + O_1 - I_1 = 360$
$I_1 + R_2 + O_2 = 430 + I_2$		$R_2 + O_2 + I_1 - I_2 = 430$
$I_2 + R_3 + O_3 = 380 + I_3$		$R_3 + O_3 + I_2 - I_3 = 380$
$I_3 + R_4 + O_4 = 450 + I_4$	or	$R_4 + O_4 + I_3 - I_4 = 430$
$I_4 + R_5 + O_5 = 520 + I_5$		$R_5 + O_5 + I_4 - I_5 = 520$
$I_5 + R_6 + O_6 = 440 + I_6$		$R_6 + O_6 + I_5 - I_6 = 440$

If the inventory variables are restricted to being nonnegative and these constraints are satisfied, then the previous demand constraints are satisfied as well. These inventory defini-tion constraints perform double duty: they define inventories, and guaranteed that demand is satisfied. Separate demand constraints are not needed.

The optimal solution for the problem is $R_1 = 370$, $R_2 = \cdots = R_6 = 420$, $O_1 = 0$, $O_2 = 0, O_3 = 0, O_4 = 10, O_5 = 80, O_6 = 20, I_1 = 10, I_2 = 0, I_3 = 40, I_4 = 20, I_5 = 0, I_6 = 0$.

■ THE GEOMETRY OF LINEAR PROGRAMS

▶ A **feasible solution** satisfies all of the constraints.

▶ **Feasible set**
Set containing all of the feasible solutions.

▶ **Optimal solution**
Feasible solution that produces the best objective function value.

The characteristic that makes linear programs easy to solve is their simple geometric structure. Let's define some terminology. A solution for a linear program is any set of numerical values for the variables. These values need not be the best values and do not even have to satisfy the constraints or make sense. For example, in the Healthy Pet Food problem, $M = 25$ and $Y = -800$ is a solution, but it does not satisfy the constraints, nor does it make physical sense. A **feasible solution** is a solution that satisfies all of the constraints. The **feasible set** or **feasible region** is the set of all feasible solutions. Finally, an **optimal solution** is the feasible solution that produces the best objective function value possible. Figure B-1 shows the relationships among these types of solutions.

Let's use the Healthy Pet Food example to show the geometry of linear programs and to show how two-variable problems can be solved graphically. The linear programming formulation for the Healthy Pet Food problem is:

$$\text{Maximize} \quad z = 0.65M + 0.45Y$$

$$\begin{aligned}
\text{Subject to} \quad 2M + 3Y &\leq 400{,}000 \\
3M + 1.5Y &\leq 300{,}000 \\
M &\leq 90{,}000 \\
M, Y &\geq 0
\end{aligned}$$
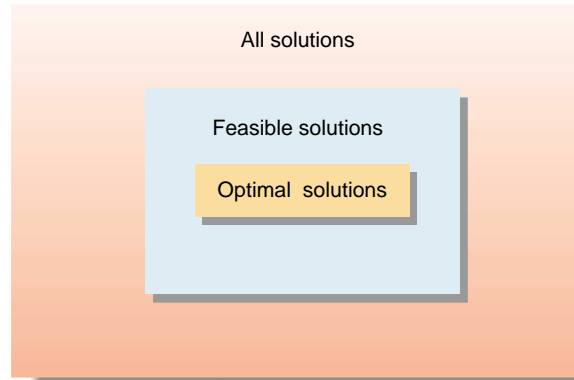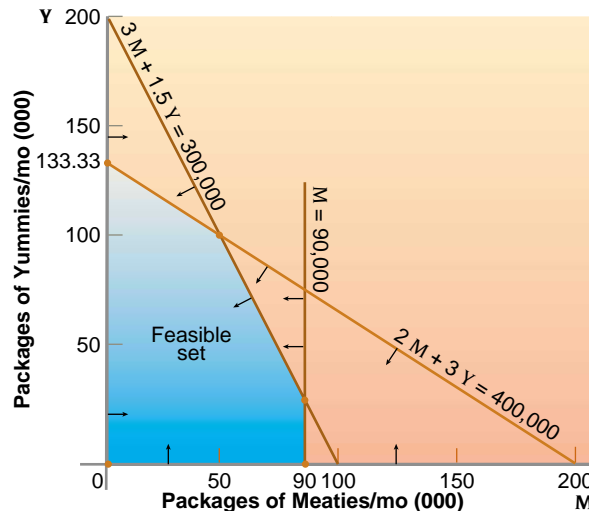
Figure B-1

Relationship among solutions

Figure B-2.

Graphical representation of the Healthy Pet Food problem

where M is the number of packages of Meaties made and sold per month and Y is the number of packages of Yummies made and sold per month. Suppose we construct a coordinate system with M measured on the horizontal axis and Y measured on the vertical axis, as shown in Figure B-2. Each point in the M, Y plane corresponds to a product mix or production plan. The coordinate values for each point represent conceivable, though not necessarily physically possible, values for the variables. Furthermore, every possible product mix is represented by a point in the M, Y plane. The best solution is the point that makes the objective function as large as possible yet satisfies all the constraints.

THE GRAPHICAL SOLUTION APPROACH ■

We begin the solution process by finding the feasible set. The geometric representation of a linear equality is the set of points that lie on and to one side of the line obtained by replacing the inequality sign with an equality sign.

The constraint $M \geq 0$ restricts us to the points on or to the right of the vertical axis (the line $M = 0$). The constraint $Y \geq 0$ restricts us to the points on or above the horizontal axis. Next, we draw the constraint $2M + 3Y \leq 400{,}000$. To find the points that satisfy this inequality, we construct the line $2M + 3Y = 400{,}000$ by finding two points that lie on the line and then constructing a line through these points. The easiest points to find on the line are the ones that lie on the two axes. First, set $M = 0$ and solve for Y. This yields the point ($M = 0$, $Y = 133{,}333.22$). We then set $Y = 0$ and solve for M. This yields the point ($M = 200{,}000$ and $Y = 0$). This line is plotted on Figure B-2.

We now determine on which side of the line the points satisfy the constraint. If one point satisfies the constraint, then all points on the same side of the line satisfy the constraint. If one point does not satisfy the constraint, then no point on that side of the line satisfies the constraint, but all the points on the opposite side of the line do satisfy the constraint. It makes sense to select a simple point with which to work, such as ($M = 0$, $Y = 0$). This point satisfies the constraint $2M + 3Y \leq 400{,}000$. Therefore, all points to the lower left do also. The points to the upper right of the line represent product mixes that require more than 400,000 pounds of cereal each month and can be eliminated from consideration.

We do the same thing for the meat constraint: $3M + 1.5Y \leq 300{,}000$. We find two points on the line $3M + 1.5Y = 300{,}000$. We first set $M = 0$ and solve for Y, and then set $Y = 0$ and solve for M, yielding the points ($M = 0$, $Y = 200{,}000$) and ($M = 100{,}000$, $Y = 0$). Checking a point on one side of the line shows that the points on or to the lower left of the line are the ones that satisfy the constraint.

The final constraint, $M \leq 90{,}000$, is satisfied by the points that lie on or to the left of the vertical line $M = 90{,}000$. The feasible set is the set of points in the five-sided shaded area in Figure B-2. The feasible set for a linear program will always have a shape like the one in this problem, with edges that are straight lines and corners where the edges meet. The corners of the feasible set are called **extreme points**. Note that each extreme point is formed by the intersection of two or more constraints.

▶ **Extreme points**
Corners of the feasible set.

The fundamental theorem of linear programming is: If a finite optimal solution exists, then at least one extreme point is optimal. To find the exact coordinate values for the optimum from the graph:

1. We identify the constraints that intersect to form the extreme point.
2. We solve simultaneously the equations corresponding to the constraints to find the point that lies on both lines (the extreme point).

In this example, the optimal extreme point is formed by the intersection of the lines $2M + 3Y = 400{,}000$ and $3M + 1.5Y = 300{,}000$.

To solve equations simultaneously, we use the following property. We can either (1) multiply any constraint by a nonzero constant or (2) add or subtract a multiple of any equation to or from any other equation, without changing the set of solutions that the equations have simultaneously.

Therefore, to solve simultaneously the equations

$$2M + \ \ 3Y = 400{,}000$$
$$3M + 1.5Y = 300{,}000$$

we can subtract two times the second equation from the first equation, leaving

$$-4M \qquad \ \ = -200{,}000$$
$$3M + 1.5Y = 300{,}000$$

The first equation is easily solved: $M = 50{,}000$. Substituting this value into the second equation gives $3(50{,}000) + 1.5Y = 300{,}000$, or $Y = 100{,}000$. This method can be used for any set of linear equations as long as the number of variables and equations is equal.

Multiple Optima, Infeasible Problems, and Unbounded Problems

Three conditions or qualifications should be noted in the fundamental theorem of linear programming. First, the theorem does not say that only extreme points can be optimal. Second, it applies only if a feasible solution exists. Third, the optimum must be finite.

Multiple Optima If a finite optimum exists for the problem, there is an extreme point that is optimal, but it may not be unique. Two or more adjacent extreme points (they share a common edge) may tie for the best solution. In this case not only are extreme points optimal, but all points on the edge connecting them are optimal. This would occur in the Healthy Pet Food example if the objective function is to maximize $z = 0.30M + 0.45Y$. Extreme points $(M = 0,\ Y = 133{,}333.33)$ and $(M = 50{,}000,\ Y = 100{,}000)$ both yield a \$60,000 profit. Every product mix on the line segment connecting these extreme points is optimal, such as $M = 20{,}000,\ Y = 120{,}000$.

Infeasible Problem In real life, we often face situations in which it is impossible to satisfy all the restrictions confronting us. For example, suppose Healthy Pet Food wanted to supply at least 160,000 packages of dog food each month; that is $M + Y \geq 160{,}000$. No points satisfy the original constraints and $M + Y \geq 160{,}000$ simultaneously. Identifying this situation is useful because we can then identify which constraints might be relaxed to obtain a feasible solution and what the consequences of relaxing the constraints will be.

Unbounded Problem Sometimes a linear program has an unbounded solution. In this situation the objective function can achieve a value of positive infinity for a maximization problem or negative infinity for a minimization problem. For example, consider the problem

Maximize	$z = A + 2B$
Subject to	$A \qquad\quad \leq 10$
	$2A + B \geq \ \ 5$
	$A, B \quad\ \ \geq \ \ 0$

As long as A is kept less than or equal to 10, B can be increased without limit and the objective function increases without limit. There is no finite optimum. Note that unboundedness refers to the objective function value, not the constraint set. It is true that for the objective function to be unbounded the feasible region must be unbounded in some direction. However, an unbounded feasible set does not imply that there is no finite optimum. To see this, we simply have to change the objective of the preceding example to minimize $A + 2B$. The feasible set is unaffected, and therefore still unbounded in some direction. However, the optimal solution is ($A = 2.5$, $B = 0$, $z = 2.5$).

Although infeasible problems can occur in practice, an unbounded problem generally indicates the modeler forgot or misrepresented one or more constraints, such as a limit on demand for a product or the supply of a resource. When an unbounded problem is encountered, the modeler should study the situation to see what limitations exist that are not being explicitly stated in the constraints.

THE SIMPLEX ALGORITHM ◼

In 1949, George Dantzig developed an efficient procedure for solving linear programs called the **simplex method** or **simplex algorithm**. This is the most widely used method in instructional and commercial computer packages. A method developed by Narendra Karmarkar in 1984 is gaining popularity, but since it requires more sophisticated mathematics, it is not presented here.

▶ **Simplex method**
Efficient method for solving linear programs.

The fundamental theorem of linear programming reduces to a finite value the number of feasible solutions that need to be evaluated. One solution strategy might be to identify the coordinates of every extreme point and then evaluate the objective function at each. The one that produces the best objective function value is the optimum. In practice, this approach is not efficient because the number of extreme points can be very large for real problems with hundreds or thousands of variables and constraints.

The simplex algorithm begins by identifying an initial extreme point of the feasible set. The algorithm then looks along each edge intersecting at the extreme point and computes the net effect on the objective function if we were to move along the edge. If the objective function value does not improve by moving along at least one of these edges, it can be proved that the extreme point is optimal. If movement along one or more of the edges improves the objective function value, we move along one of these edges until we reach a new extreme point. We repeat the previous steps: checking along each edge intersecting at the extreme point, and then either stopping or sliding along another edge that improves the objective function value.

This algorithm has many desirable features in practice.

1. It only moves from one extreme point to a better or equally good extreme point, thereby skipping large numbers of suboptimal extreme points without explicitly identifying them. Thus it usually only has to check a small subset of the extreme points to find an optimum.
2. When it finds an optimum, it identifies this fact and stops.
3. The algorithm detects whether the problem is infeasible, is unbounded, or has multiple optima.
4. The algorithm uses very simple mathematics that are easy to implement on a computer.

Preparing the Problem for Solution

▶ **Canonical form** is needed to use the simplex method.

To use the simplex algorithm, we write the problem in **canonical form**. Four conditions must be satisfied for the problem to be in canonical form.

1. The right-hand sides of all functional constraints (the constraints other than nonnegativity of the variables) must be nonnegative. Zero is an acceptable right-hand-side value.
2. All constraints must be written as equalities.
3. All variables must have a ≥ 0 restriction.
4. Every functional constraint must contain a variable that appears only in that constraint and not in any other constraint. That is, the variable has a zero coefficient in all other functional constraints. It must have a $+1$ coefficient in the constraint in which it appears.

The following shows how to convert a linear program into canonical form.

Nonnegative Right-Hand Sides. Suppose a constraint has a negative right-hand side, such as

$$3x_1 + 4x_2 - 2x_3 \leq -6$$

The right-hand side can be made nonnegative by multiplying through the entire constraint by -1, so the constraint becomes

$$-3x_1 - 4x_2 + 2x_3 \geq 6$$

Note that the direction of the inequality reverses when the constraint is multiplied by a negative constant.

All Constraints Must Be Equalities. Any inequality constraint can be rewritten as an equivalent equality by introducing a nonnegative variable called a slack variable. Let's look at $\leq$ constraints first.

$$3x_1 + 4x_2 \leq 12$$

If we define a new variable, s_1, as $s_1 = 12 - (3x_1 + 4x_2)$, then this constraint can be written as

$$3x_1 + 4x_2 + s_1 = 12$$

For any solution, the variable s_1 measures the "slack" that exists in the constraint. That is, s_1 measures by how much a solution satisfies a constraint.

We use a similar approach to convert $\geq$ constraints into equalities. For example, consider

$$2x_1 + 4x_2 \geq 10$$

If we define a new variable, s_2, so that $s_2 = (2x_1 + 4x_2) - 10$, then we can rewrite the constraint as

$$2x_1 + 4x_2 - s_2 = 10$$
$$s_2 \geq 0$$

In summary, for $\leq$ constraints, we add a nonnegative slack variable to the left-hand side of the constraint, and for $\geq$ constraints we subtract a nonnegative slack variable from the left-hand side of the constraint.

Slack variables have physical meaning. For example, slack variables may measure how much of an available resource is not used for a given solution or how much of a nutrient an animal receives above some minimum required amount. The slack variables in each constraint measure something different. Therefore, each one should have a different designation. A standard convention uses s_i for a slack variable that has been incorporated into constraint i.

All Variables Must Be Nonnegative. Most linear programming computer packages automatically convert right-hand-side constants into nonnegative values and change constraints into equalities. They assume all variables are nonnegative. If they are not, the problem must be rewritten so that all variables are nonnegative.

Each Constraint Must Have a Unique Variable with a + 1 Coefficient. Using the Healthy Pet Food problem, we perform the needed steps, giving us the following:

$$\text{Maximize} \qquad z = 0.65M + 0.45Y$$

$$
\begin{aligned}
\text{Subject to} \qquad 2M + 3Y + s_1 \qquad\quad &= 400{,}000 \\
3M + 1.5Y \quad + s_2 \quad\; &= 300{,}000 \\
M \qquad\qquad\quad + s_3 &= 90{,}000
\end{aligned}
$$

All variables ≥ 0

We check to see whether each constraint has a variable that appears only in that constraint and has a coefficient of $+1$. In the first constraint, s_1 satisfies this requirement; in the second constraint, s_2 satisfies it; and in the third constraint, s_3 satisfies it. This problem is in canonical form and ready to solve.

The Algebraic Foundations of the Algorithm

In the initial formulation the number of constraints may exceed the number of variables. Once the problem is in canonical form, the number of variables k should always be at least as large as the number of functional constraints m. In the Healthy Pet Food problem, $k = 5$ and $m = 3$. Because the number of variables exceeds the number of constraints, there are an infinite number of solutions that satisfy these equations. We find a solution to these equations by setting the values of $k - m$ of the variables ($5 - 3 = 2$ in this case) equal to zero and then solving the m equations simultaneously to obtain the values of the remaining m (three) variables. Any solution obtained using this procedure is called a basic solution. In a basic solution the $k - m$ variables set equal to zero are called the **nonbasic variables** for that solution. The remaining m variables, the values that are obtained by solving the m equations, are called the **basic variables**. Note that basic variables can equal zero. Nonbasic variables must equal zero. The set of basic variables for a **basic solution** is called the basis for that solution. The basic variables are said to be "in the basis" and the nonbasic variables are "not in the basis."

Suppose that after we put the Healthy Pet Food problem into canonical form, we choose M and Y as our nonbasic variables and set them equal to zero. The function constraints reduce to

$$
\begin{aligned}
s_1 &= 400{,}000 \\
s_2 &= 300{,}000 \\
s_3 &= 90{,}000
\end{aligned}
$$

▶ **Nonbasic variables**
Variables set equal to zero.

▶ **Basic variables** have values obtained by solving the m equations.

▶ **Basic solution**
Current set of basic variables.

This system of equations is trivial to solve. Once the problem is in canonical form, a basic solution jumps out at us. The unique variable in each constraint that has a coefficient of $+1$ will be the basic variable for that constraint. Not only are there exactly m basic variables, but there is also a one-to-one correspondence between each constraint and the basic variables. Any variable that is not basic is nonbasic and is set equal to zero. This reduces the m equations to a form in which each equation has only one basic variable in it. Therefore, that variable must equal the value on the right-hand side.

Basic solutions represent the points where constraints intersect. Not all basic solutions are feasible. When a basic solution is feasible, it is called a basic feasible solution. Basic feasible solutions are the algebraic representation of extreme points. We know from the fundamental theorem of linear programming that if a finite optimum exists, an extreme point will be optimal.

The basic feasible solution ($M = 0$, $Y = 0$, $s_1 = 400{,}000$, $s_2 = 300{,}000$, $s_3 = 90{,}000$) has an objective function value of 0. We want to determine whether this is an optimal solution. We look along each constraint that intersects at this extreme point. If the objective function improves by moving along one of the edges, the current solution is not optimal. If there is no edge along which the objective function improves, the current solution is optimal.

Moving along the edge is accomplished algebraically by increasing the value of one of the nonbasic variables while keeping all other nonbasic variables equal to zero and then adjusting the values of the basic variables to maintain feasibility. For example, suppose we increase the value of M from 0 to 1, keep $Y = 0$, and adjust the values of the remaining variables. If M increases by one unit, then s_1 must decrease by two units to 399,998 to keep the first equation satisfied; and s_2 must decrease by three units to 299,997, while s_3 must decrease by one unit to 89,999. Geometrically, we slid one unit along the M axis (the edge of the $Y \geq 0$ constraint) as shown in Figure B-3. Note that this is not a basic solution. A basic solution has at least two variables equal to zero and represents a point where constraints intersect.

The objective function value increases by 0.65 by moving one unit along this edge. Each unit we move in this direction causes the objective function to increase by 0.65. Therefore, we want to increase M as much as possible. However, as M increases the previous basic variables all decrease. At some point one of them will be forced to zero. Specifically, for every unit that M increases, s_1 decreases by two units. Because s_1

Figure B-3.

Geometry of a simplex pivot
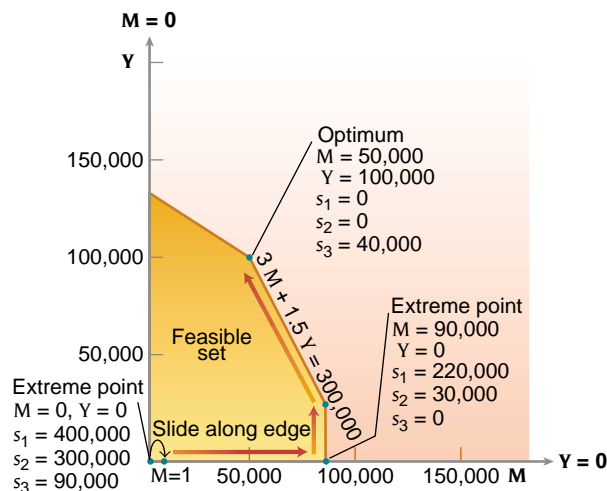
begins at 400,000, M can increase by 200,000 units before s_1 hits zero. Similarly, M can increase by 100,000 units before s_2 is forced to zero, and M can increase by 90,000 before s_3 reaches zero. Therefore, the most we can increase M before one of the basic variables is forced to zero is 90,000 units. If M increases beyond this point, s_3 becomes negative and the solution is infeasible.

By increasing M to 90,000, keeping $Y = 0$, and adjusting the other variables, we obtain a new (adjacent) basic feasible solution: $M = 90,000$, $Y = 0$, $s_1 = 220,000$, $s_2 = 30,000$, and $s_3 = 0$, where Y and s_3 are the nonbasic variables.

The Initial Simplex Tableau

To use the simplex algorithm, we write the problem in a more efficient form using **simplex tableaus**. In a simplex tableau we omit the variable names and work only with the relevant coefficients. Table B-4 gives the initial simplex tableau for the Healthy Pet Food problem.

▶ **Simplex tableau**
More efficient form of the problem data.

In the tableau, a column is assigned to each variable. We write the coefficients that apply to each variable by placing the coefficients in the appropriate columns. This eliminates the need to write the variable names repeatedly. The top row of numbers is called the *objective function* or c_j row. The objective function coefficients for each variable are written in this row, including coefficients that are zero. The next rows contain the coefficients of the constraints. The elements in these rows are called the a_{ij} values. The subscript i designates the number of the constraint and the j identifies which variable it is multiplying. For example, $a_{21} = 3$ is the coefficient of the second constraint and it multiplies the first variable, M. The right-most column is called the *right-hand side* or b *column*. The element in the ith row of this column will be called b_i. In the second column from the left we list the current basic variable for each constraint. The basic variable for the ith constraint will have a coefficient of $+1$ in that constraint and a coefficient of 0 in every other constraint. The leftmost column, called the c_B column, lists the objective function coefficients for the basic variables. The element in the ith row of this column will be designated c_{Bi}.

The last two rows of the tableau are used to determine whether the current basic feasible solution is optimal, and if not, which variable should enter the basis. The second row from the bottom is the z_j *row*. For the jth variable,

$$z_j = \sum_{i=1}^{m} a_{ij} c_{Bi}$$

Table B-4 First Healthy Pet Food Tableau

c_B	Basic Variable	M 0.65	Y 0.45	s_1 0	s_2 0	s_3 0	b	Ratio	
0	s_1	2	3	1	0	0	400,000	(400,000/2) = 200,000	
0	s_2	3	1.5	0	1	0	300,000	(300,000/3) = 100,000	
0	s_3	1	0	0	0	1	90,000	(90,000/1) = 90,000	←Pivot row
	z_j	0	0	0	0	0	0		
	$c_j - z_j$	0.65	0.45	0	0	0			

↑
Pivot
column

Thus to compute z_j for the jth variable, we go down the jth variable column and the c_B column, performing pairwise multiplication and then summing the terms. For example, in Table B-4, $z_1 = (2 \times 0) + (3 \times 0) + (1 \times 0) = 0$. We perform this operation for every variable and for the right-hand side values: $(400{,}000 \times 0) + (300{,}000 \times 0) + (90{,}000 \times 0) = 0$. The z_j in the b column is the objective value for the basic solution in the tableau. We explain the physical meaning of the z_j values shortly.

The bottom row is called the $c_j - z_j$ *row*. It represents exactly that: we subtract the z_j row from the c_j row. There is no value in the b column because there is no b column value in the c_j row.

Meaning of z_j and $c_j - z_j$

The algorithm moves from one basic solution to another by

1. Increasing the value of a nonbasic variable
2. Keeping the other nonbasic variables equal to zero
3. Adjusting the values of the basic variables to satisfy the functional constraints

When this is done, two forces affect the objective function value. There is the direct effect of the nonbasic variable that is being increased in value. The per unit or marginal effect of increasing the jth variable is the objective function coefficient for this variable, c_j. When the basic variables are adjusted to maintain feasibility, however, this adjustment affects the objective function as well. This adjustment effect, often called the opportunity cost, is measured by z_j. The opportunity cost is the penalty resulting from adjusting these basic variables. The net marginal effect on the objective function is $c_j - z_j$. If the $c_j - z_j$ for some nonbasic variable is positive in sign, increasing that variable (and adjusting the other variables to remain feasible) results in a net increase in the objective function value.

The Simplex Pivot and the Second Tableau

Once the problem is set up in the initial tableau, the algorithm uses the following four-step procedure.

1. *Check the current basic solution for optimality.* For a maximization problem, if all the $c_j - z_j$ values are zero or negative, increasing the value of any variable will not increase the objective function value. Therefore the current solution is optimal. For a minimization problem, we stop when all of the $c_j - z_j$ values are zero or positive. This means that increasing the value of any variable will not decrease the objective function value. Note that the $c_j - z_j$ values for the basic variables in a tableau always equal zero.

2. *Select the entering variable.* In Table B-4, the $c_j - z_j$ values for both M and Y are strictly positive. This means that bringing either of these variables into the basis increases the value of the objective function. There is no way to know in advance which variable will lead us to the optimum the quickest. A simple rule that works well is to select the variable that has the most positive $c_j - z_j$ if we are maximizing, or the most negative $c_j - z_j$ if we are minimizing. In our example, we select M to enter the basis. This is called the entering variable. The M column is called the **pivot column**.

3. *Identify the leaving variable.* As shown above, as we increase the value of M while keeping Y fixed at zero, the other basic variables must change in value to satisfy the constraints. As M increases, s_1, s_2, and s_3 all decrease in value. Variable s_3 reaches zero first and leaves the basis. The leaving variable can be

▶ **The pivot column** has the most positive $c_j - z_j$ value if we are maximizing or the most negative $c_j - z_j$ value if we are minimizing.

determined by using a **ratio test**. For every constraint row i we compute the ratio b_i/a_{is}, if $a_{is} > 0$, where column s is the pivot column. That is, we divide the right-hand side of each constraint by the element in the pivot column of the same row, but only if the denominator a_{is} is strictly positive in value. This ratio computes how much the entering variable can increase in value before the basic variable in that constraint is forced down to zero. That leaves the basic variable in the row with the smallest ratio. This row is called the **pivot row**. In Table B-4 the ratios for the three constraints are $[400{,}000/2] = 200{,}000$, $[300{,}000/3] = 100{,}000$, and $[90{,}000/1] = 90{,}000$. The smallest ratio is for the third constraint, so M enters the basis as the variable for the third constraint and forces s_3 out of the basis.

▶ **The ratio test** is used to determine the leaving variable.

▶ **Pivot row**
The row with smallest ratio.

4. *Perform pivot operations to obtain the new canonical form.* The intersection of the pivot row and pivot column is called the **pivot element**. We need to rewrite the constraints so that they are mathematically equivalent and make sure that the new basic variable, M, appears in only the third constraint and has a coefficient of $+1$. In other words, we want the M column to contain $+1$ in the pivot element and 0 in every other constraint row. We do this with row operations. The problem does not change if we multiply any equation by a nonzero constant, or add or subtract a multiple of any equation to or from any other equation. We divide the entire pivot row by the value of the pivot element. In Table B-4, the pivot element is already equal to 1, so the pivot row does not change. It is just rewritten in Table B-5. If the pivot element had been 4, we would divide the entire pivot row (except the two leftmost columns) by 4. The new form of the pivot row is always a multiple of the current pivot row, so a multiple of this new form of the pivot row can be added to or subtracted from the other rows.

▶ **Pivot element**
Intersection of the pivot row and the pivot column.

To obtain a zero in the M column of the first constraint, we multiply the pivot row (third row) by 2 and subtract from the first constraint row:

$$[2 \quad 3 \quad 1 \quad 0 \quad 0 \quad 400{,}000]$$
$$-2 \times [1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 90{,}000] = [0 \quad 3 \quad 1 \quad 0 \quad -2 \quad 220{,}000]$$

The new form of the first constraint in shown in Table B-5. To obtain a zero in the M column of the second constraint, multiply the pivot row by 3 and subtract from the second constraint row:

$$[3 \quad 1.5 \quad 0 \quad 1 \quad 0 \quad 300{,}000]$$
$$-3 \times [1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 90{,}000] = [0 \quad 1.5 \quad 0 \quad 1 \quad -3 \quad 30{,}000]$$

	Table B-5	Second	Healthy	Pet	Food	Tableau		
c_B	**Basic Variable**	**M** 0.65	**Y** 0.45	**s_1** 0	**s_2** 0	**s_3** 0	**b**	**Ratio**
0	s_1	0	3	1	0	−2	220,000	(220,000/3) = 73,333
0	s_2	0	1.5	0	1	−3	30,000	(30,000/1.5) = 20,000 ←Pivot row
0.65	M	1	0	0	0	1	90,000	no ratio
	z_j	0.65	0	0	0	0.65	58,500	
	$c_j - z_j$	0	0.45	0	0	−0.65		

↑
Pivot
column

The result is shown in Table B-5. Note that we always use the pivot row to operate on the other rows. If we add or subtract multiples of a row other than the pivot row to or from a row, the columns for the other basic variables can become messed up. Also note that because the new form of the pivot row always has a 1 in the pivot element, we create a 0 in pivot column for any other constraint just by multiplying the new form of the pivot row by the element in the pivot column of the other row and subtracting the result from that row.

At this point, we determine the basic solution for the tableau in Table B-5. Y and s_3 are not in the basis, so they are set equal to zero. Then, $s_1 = 220{,}000$, $s_2 = 30{,}000$, and $M = 90{,}000$. We compute z_j and $c_j - z_j$ elements for each column. For example, $z_2 = (3 \times 0) + (1.5 \times 0) + (0 \times 0.65) = 0$. Note that the z_j on the right-hand side equals the objective function value, 58,500. We will call this term z from now on.

The Third Tableau

The solution in Table B-5 is not optimal because $c_2 - z_2 = 0.45 > 0$. Therefore, we select Y as our entering variable. The leaving variable is determined by performing the ratio test. In this case, the second constraint has the smallest ratio (row 3 has no ratio because $a_{32} = 0$), so variable s_2 leaves the basis.

To obtain the new tableau, we divide the pivot row by the pivot element, 1.5, and write the result in Table B-6. The third row already has a zero in the pivot column, so we just rewrite that constraint. To obtain a zero in the pivot column of the first constraint row, we multiply the new form of the pivot row in Table B-6 by 3 and subtract from the first constraint row in Table B-5. The new tableau is given in Table B-6, where the basic solution is $M = 90{,}000$, $Y = 20{,}000$, $s_1 = 160{,}000$, $s_2 = 0$, $s_3 = 0$, and $z = \$67{,}500$.

The Fourth Tableau

The solution in Table B-6 is not optimal because $z_5 = 0.25 > 0$, so we bring s_3 back into the basis. Note, first, that a variable can leave the basis and return to it later. Second, even though the direct benefit of increasing the value of s_3 is zero, this tableau shows that the objective function increases due to the adjustment of the other basic variables, which is measured by z_5.

By performing the ratio test, we find that s_1 leaves the basis. The new tableau is given in Table B-7. The resulting basic solution is $M = 50{,}000$, $Y = 100{,}000$, $s_1 = 0$, $s_2 = 0$, $s_3 = 40{,}000$, and $z = \$77{,}500$. This solution is optimal because all of the $c_j - z_j$ values are less than or equal to zero.

Table B-6 Third Healthy Pet Food Tableau

c_B	Basic Variable	M 0.65	Y 0.45	s_1 0	s_2 0	s_3 0	b	Ratio	
0	s_1	0	0	1	−2	4	160,000	(160,000/4) = 40,000	←Pivot row
0.45	Y	0	1	0	2/3	−2	20,000	no ratio	
0.65	M	1	0	0	0	1	90,000	(90,000/1) = 90,000	
	z_j	0.65	0.45	0	0.30	−0.25	67,500		
	$c_j - z_j$	0	0	0	−0.30	0.25			

↑
Pivot
column

c_B	Basic Variable	M 0.65	Y 0.45	s_1 0	s_2 0	s_3 0	b
	Table B-7 Fourth Healthy Pet Food Tableau						
0	s_3	0	0	1/4	−1/2	1	40,000
0.45	Y	0	1	1/2	−1/3	0	100,000
0.65	M	1	0	−1/4	1/2	0	50,000
	z_j	0.65	0.45	0.0625	0.175	0	77,500
	$c_j - z_j$	0	0	−0.0625	−0.175	0	

General Comments about the Algorithm

1. The right-hand sides of the constraints in each tableau should always be nonnegative. A negative value on the right-hand side can have two causes: (a) the pivot row was chosen incorrectly (a row other than the one with the smallest ratio was used) or (b) an arithmetic error was made.
2. The $c_j - z_j$ values for the basic variables should always be zero.
3. The objective function value should never get worse in going from one tableau to the next. In fact, it should improve by the product of $c_j - z_j$ for the entering variable and the entering variable's value in the new tableau. For example, in Table B-6, $c_5 - z_5 = 0.25$, so for each unit s_3 is increased, the objective function value should increase by $+0.25$. Going from the solution in Table B-6 to the one in Table B-7, s_3 increased by 40,000 units, so the objective function value increases by $40,000 \times \$0.25 = \$10,000$, which is exactly what happened.

USING ARTIFICIAL VARIABLES ■

In the Healthy Pet Food problem all of the constraints were $\leq$ constraints. Thus as soon as slack variables are added to the left-hand side, they satisfy the canonical condition for the constraints and are the initial basic variables. If a linear program has $=$ or $\geq$ constraints this does not occur. There is no basic variable readily at hand for an initial basis. Consider the problem

Maximize $\qquad\qquad\qquad z = 20A + 6B + 9C$

Subject to
$$4A + 3B + C \leq 24$$
$$2A + 4B + 12C \geq 30$$
$$2A + 3B + C = 10$$

All variables ≥ 0

After adding the appropriate slack variables, the problem is

Maximize $\qquad\qquad\qquad z = 20A + 6B + 9C$

Subject to
$$4A + 3B + C + s_1 = 24$$
$$2A + 4B + 12C - s_2 = 30$$
$$2A + 3B + C = 10$$

All variables ≥ 0

The variable s_1 appears only in the first constraint and has a coefficient of $+1$, so it can be the basic variable for the first constraint. Although s_2 appears only in the

second constraint, it has a coefficient of -1; therefore, there is no variable in the second constraint that can act as a basis variable (multiplying the second constraint by -1 makes the right-hand-side coefficient negative). The third constraint also has no unique variable ready to be basic. In this case we create a basic but infeasible solution by adding two new nonnegative variables, A_2 A_3, to the left-hand side for the second and third constraints. These are called **artificial variables**. (We use the notation A_i where it is added to the *i*th constraint.) This gives the new form of the problem, which is in canonical form.

▶ **Artificial variables** are needed when $=$ or $\geq$ constraints appear in the problem.

Maximize $\qquad\qquad\qquad\qquad\qquad z = 20A + 6B + 9C$

Subject to $\qquad\qquad\qquad\qquad 4A + 3B + C + s_1 \qquad\qquad\quad = 24$
$\qquad\qquad\qquad\qquad\qquad\quad 2A + 4B + 12C - s_2 + A_2 \qquad = 30$
$\qquad\qquad\qquad\qquad\qquad\quad 2A + 3B + C \qquad\qquad\quad + A_3 = 10$

All variables ≥ 0

The artificial variables are put into the problem for only one purpose: *to act as basic variables to the algorithm started*. If an artificial variable is in the problem, it must be a basic variable. In the current canonical form the basic variables are s_1, A_2, and A_3, and the corresponding basic solution is $A = B = C = s_2 = 0$, $s_1 = 24$, $A_2 = 30$, and $A_3 = 10$. This is a basic solution that occurs at the intersection of the constraint boundaries: $A = 0$, $B = 0$, $C = 0$. However, this solution is infeasible because these lines intersect outside the feasible set.

By modifying the objective function to "encourage" the artificial variables to leave the basis, we use the simplex algorithm to move from an infeasible to a feasible basic solution. The algorithm then proceeds exactly as in the previous section. There are two ways to modify the problem. The first, called the *Big-M method*, is intuitively simple and is demonstrated here.

The Big-*M* Method

▶ **Big-*M* method** uses objective function coefficients for the artificial variables of $-$infinity for maximization problems or $+$infinity for minimization problems.

In the **Big-*M* method,** we make the objective function coefficients for the artificial variables either $-$infinity for maximization problems or $+$infinity for minimization problems. The name comes from the fact that the letter *M* represents some large number that replaces infinity. This method encourages the algorithm to drive the artificial variables to zero by kicking them out of the basis. Rather than use $-$infinity, in our example $M = -1000$. The first tableau is in Table B-8.

Table B-8 First Tableau: Big-*M* Method

c_B	Basic Variable	A 20	B 6	C 9	s_1 0	s_2 0	A_c -1000	A_B -1000	b	Ratio	
0	s_3	4	3	1	1	0	0	0	24	(24/1) = 24	
$-1,000$	A_2	2	4	12	0	-1	1	0	30	(30/12) = 2.5	←Pivot row
$-1,000$	A_3	2	3	1	0	0	0	1	10	(10/1) = 10	
	z_j	$-4,000$	$-7,000$	$-13,000$	0	1,000	$-1,000$	$-1,000$	$-4,000$		
	$c_j - z_j$	4,020	7,006	13,009	0	$-1,000$	0	0			

↑
Pivot column

Variable C has the most positive $c_j - z_j$, so C enters the basis. The second row has the smallest ratio, so the basis variable in that row, A_2, will leave the basis. Performing the pivot yields Table B-9. Note that the A_2 column has been deleted in Table B-9. As soon as an artificial variable leaves the basis, it is dropped from the problem. (*Do not confuse this with other types of variables.*) The resulting solution is not optimal so variable B enters the basis and replaces A_3, giving Table B-10. Variable A then enters the basis and B leaves, giving Table B-11.

The optimal solution is $A = 45/11 = 4.091$, $B = 0$, $C = 20/11 = 1.818$, $s_1 = 64/11 = 5.818$, $s_2 = 0$, and $z = 1080/11 = 98.181$. Although the artificial variables usually leave the basis quickly, they are not always the first variables to leave as they were in this example.

Table B-9 Second Tableau: Big-*M* Method

c_B	Basic Variable	A 20	B 6	C 9	s_1 0	s_2 0	A_c −1000	A_B −1000	b	Ratio
0	s_3	23/6	8/3	0	1	1/12	—	0	21.5	(21.5/(8/3)) = 8.06
9	C	1/6	1/3	1	0	−1/12	—	0	2.5	(2.5/(1/3)) = 7.5
−1000	A_3	11/6	8/3	0	0	1/12	—	1	7.5	(7.5/(8/3)) = 2.81 ←Pivot row
	z_j	−1831.8	−2663.7	9	0	−84.1	—	−1000	−7477.5	
	$c_j - z_j$	1851.8	2669.7	0	0	84.1	—	0		

↑ Pivot column

Table B-10 Third Tableau: Big-*M* Method

c_B	Basic Variable	A 20	B 6	C 9	s_1 0	s_2 0	A_c −1000	A_B −1000	b	Ratio
0	s_3	2	0	0	1	0	—	—	14	(14/2) = 7
9	C	−1/16	0	1	0	−3/32	—	—	25/16	no ratio
6	B	11/16	1	0	0	1/32	—	—	45/16	(45/11) = 4.09 ←Pivot row
	z_j	57/16	6	9	0	−21/32	—	—	495/16	
	$c_j - z_j$	263/16	0	0	0	21/32	—	—		

↑ Pivot column

Table B-11 Optimal Tableau: Big-*M* Method

c_B	Basic Variable	A 20	B 6	C 9	s_1 0	s_2 0	A_2 −1000	A_3 −1000	b
0	s_3	0	−32/11	0	1	−1/11	—	—	64/11
9	C	0	1/11	1	0	−1/11	—	—	20/11
20	A	1	16/11	0	0	1/22	—	—	45/11
	z_j	20	329/11	9	0	1/11	—	—	1080/11
	$c_j - z_j$	0	−263/11	0	0	−1/11	—	—	

The Two-Phase Method

For the Big-M method to work well, the value of M must be several orders of magnitude larger than the other objective function coefficients. This large difference in coefficient magnitude can cause numerical instability and roundoff problems in computer programs. An alternative used more widely in practice, the two-phase method, accomplishes the same thing but is numerically more stable. The **two-phase method** divides the solution process into two phases. In phase 1, we want to move from the initial basic infeasible solution to some basic feasible solution. In phase 2, we want to move to the best basic feasible solution. The goal of phase 1 is accomplished by replacing the original objective function with the objectives of minimizing the sum of the artificial variables. In other words, the c_j values for all variables are 0 except for the artificial variables, which have c_j values of 1. This is the phase 1 objective function regardless of whether the original objective is to maximize or minimize. This objective function causes the artificial variables to go to zero by leaving the basis. As soon as all the artificial variables are out of the basis, phase 2 begins. In phase 2, the original objective function for the problem is used, and the solution proceeds as before.

Infeasible Problems, Multiple Optima, Unboundedness, and Degeneracy and the Simplex Algorithm

One feature of the simplex algorithm is that it identifies infeasible problems, multiple optima, and unboundedness if they exist.

Infeasible Problem If for some tableau the $c_j - z_j$ values satisfy the optimality conditions but at least one artificial variable is still positive (in the basis), then the problem has no feasible solution.

Multiple Optima If the problem has more than one optimum, this is signaled by the final tableau. In the optimal tableau, if $c_j - z_j$ for a nonbasic variable is zero, that variable entering into the basis does not change the objective function value, so the new solution is also optimal.

Unbounded Problem If for any tableau $c_j - z_j$ indicates that a nonbasic variable should enter the basis, but no ratios can be computed for the constraints because every constraint coefficient in the pivot column is either zero or negative, the problem is unbounded. Increasing the value of the entering variable improves the objective function without limit.

Degeneracy The right-hand sides of a tableau should never be negative, but they can be zero. When a right-hand side is zero, a basic variable equals zero; this is a **degenerate solution**. You treat the zero just like any other number. It can be used as the numerator in the ratio test. Degeneracy simply means that the extreme point represented by the basic solution is formed by the intersection of more constraints than are needed. Therefore, the same extreme point can be expressed algebraically by several basic solutions.

COMPUTER SOLUTIONS OF LINEAR PROGRAMS ■

Once a linear program is formulated, it is solved using a computer-based solution method. Most commercial and instructional linear programming computer packages use methods based on the simplex algorithm. Many computer packages are available for personal computers. Although each package is slightly different, the input information required and the output provided by typical instructional packages are similar.

Problem Input

Most linear programming packages use the following information to set up the problem:

1. Whether the user wants to enter a new problem, read an existing problem, modify the current problem in memory, or solve a problem
2. An identifying name for the problem so that it can be stored and reused or modified at a future time
3. Whether the problem involves maximization or minimization
4. The number of variables in the problem
5. The names of the variables or whether the user wants to use default names, usually X1, X2, and so on
6. The number of constraints in the problem (excluding nonnegativity constraints on the variables)
7. The objective function and constraint coefficients. These can be entered as a data matrix, or a problem template of the objective function constraints is displayed with variables listed and spaces provided for the user to enter the coefficients.

Computer Output

Most packages offer several output options: to display the optimum, the optimum with sensitivity analysis, the initial simplex tableau and the optimum, and so forth. A computer output for the Healthy Pet Food problem is shown in Figure B-4. Part (a) of the figure lists the optimal values for the variables in the column labeled "solution": $M = 50{,}000$, $Y = 100{,}000$, S1 $= 0$, S2 $= 0$, and W3 $= 40{,}000$. The slack variables are added to the problem automatically by the computer package, and their optimal values are listed. At the bottom of Figure B-4 is the objective function value: $77,500.

Sensitivity Analysis

An important use of linear programs is in determining how sensitive the optimal solution is to parameter values in the problem. For example, how will the optimum change if a price of availability of a resource is changed? This is called sensitivity analysis or parametric analysis. Let's look at the sensitivity analysis information provided by linear programming computer packages.

Changes in Objective Function Coefficients If an objective function coefficient is changed, this does not affect the feasible set of solutions for the problem. Figure B-4 (b), labeled "Sensitivity Analysis for Objective Function Coefficients," lists the

Figure B-4.

Computer solution for Healthy Pet Food

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(a) Final Solution for Healthy

Variable	Solution	Opportunity Cost
M	+50000.00	0
Y	+100000.00	0
S1	0	+0.0625
S2	0	+0.1725
S3	+40000.00	0

Maximum Objective Function Value = 77500
Iterations = 3
CPU time = 0.4215

(b) Sensitivity Analysis for Objective Function Coefficients

Variable	Min C(j)	Original	Max C(j)
M	+0.3000	+0.6500	+0.9000
Y	+0.3250	+0.4500	+0.9750

(c) Sensitivity Analysis for Right-hand-side Coefficients

Constraint	Min B(i)	Original	Max B(i)	Shadow Price
1	+240000	+400000	+600000	+0.0625
2	+200000	+300000	+380000	+0.1750
3	+ 50000	+ 90000	+ Infinity	0

</div>

range over which each objective function coefficient can vary without the optimal values for the variables changing.

The notation C(j) refers to the objective function coefficient for the *j*th variable. The values in the column labeled "Original" are the original objective function coefficients in the problem: 0.65 for M and 0.45 for Y. In the M row, the values 0.30 in the "Min C(j)" column and 0.90 in the "Max C(j)" column mean that as long as all the other data in the problem are held constant, the solution M = 50,000 and Y = 100,000 is optimal for any value of C(1) from a minimum of 0.30 to a maximum of 0.90. For example, suppose the profit per package of Meaties changes from $0.65 to $0.85. The information in Figure B-4 (b) implies that M = 50,000 and Y = 100,000 is still the optimal solution. However, the optimal amount of profit changes to $87,500.

One limitation of this analysis is that the range of coefficient values for which the solution remains optimal is valid only when one coefficient is modified. If two or more coefficients are changed simultaneously, the ranges given in Figure B-4 (b) do not tell us what happens. In this case, we change the coefficient values and re-solve the problem.

Dual Prices and Changes in the Right-Hand-Side Coefficients

Suppose the right-hand side of a constraint is changed. For example, in the Healthy Pet Food problem, suppose Healthy can buy up to 400,001 pounds of cereal each month (at $0.20 per pound) instead of 400,000 pounds. Will this change the optimal solution, and if so, how? The information in Figure B-4(c) partially answers this question. For each constraint there is a **shadow price** or **dual price**. The dual price for constraint 1 is 0.0625, and the dual prices for constraints 2 and 3 are 0.1750 and 0, respectively.

Each constraint has a dual price. The dual price for a constraint is the marginal change in the optimal objective function value that occurs if the right-hand side of

▶ **Shadow price**
The marginal change in the optimal objective function value that occurs if the right-hand side of a constraint is changed.

that constraint is altered. For example, dual price 0.0625 for constraint 1 implies that if one addition pound of cereal is available (at $0.20 per pound), the optimum production plan can be modified to produce $0.0625 more profit. Similarly, the dual price of $0.175 means that if an additional pound is available (at $0.50 per pound), the optimum can be modified to produce $0.175 more profit. It must be emphasized that these dual prices are valid only if one right-hand side is modified. If two or more changes are made simultaneously, these prices may not be valid.

Figure B-4(c) also gives the range of right-hand-side values over which the dual prices will be valid. For example, B(1) is the right-hand-side coefficient of the first constraint. The table says that keeping everything else the same, each additional pound of cereal increases profit by $0.0625 until the right-hand side equals 600,000 (shown in the "Max B (I)" column. Similarly, a reduction in cereal reduces the profit by $0.0625 per pound until the right-hand side reaches 240,000 pounds ("Min B (i)"). If the amount of available cereal is changed enough to be outside these limits, new dual prices apply. This analysis is sometimes called right-hand-side ranging.

USING LINEAR PROGRAMMING MODELS FOR DECISION MAKING ■

The formulations earlier in the supplement give the impression that using linear programming is a clean, simple process. We recognize a problem that fits the linear programming framework, model it, solve it, and then we are done. In practice, using linear programming and other optimization models is not so straightforward, nor is it static. Specifically, our goal in using models is to obtain usable solutions that are better than those we would have obtained without the models, to use the models to revise and update our decisions in a timely fashion, and to increase our confidence in our decisions. Let's look at the Healthy Pet Food problem to see how we might use his model in practice and to see what benefits we can gain.

Healthy Pet Food Revisited

The Healthy Pet Food example was presented as if every aspect of the problem were known with certainty, and a number of simplifying assumptions were made. Let's reconsider the situation to show more completely how linear programming models can be used. Refer to Table B.1 for the original problem data.

The linear programming formulation for this problem is:

Maximize $\qquad z = 0.65M + 0.45Y$

Subject to
$$
\begin{aligned}
2M + 3Y + s_1 &= 400{,}000 \text{ (cereal constraint)} \\
3M + 1.5Y + s_2 &= 300{,}000 \text{ (meat constraint)} \\
M + s_3 &= 90{,}000 \text{ (Meaties capacity)}
\end{aligned}
$$

All variables ≥ 0

and the optimal solution is $M = 50{,}000$, $Y = 100{,}000$, and $z = \$77{,}500$ per month. This solution causes Healthy to use all 400,000 pounds of cereal and all 300,000 pounds of meat available each month at $0.20 and $0.50 per pound, respectively.

Uncertainty Regarding Price and Cost Data   We assumed that the data given in Table B-1 were known with certainty. In fact, there is considerable uncertainty about the price and cost estimates. Suppose that after seeing the preceding solution, the vice president of marketing says, "If we're only going to produce 50,000 packages

of Meaties each month, we can charge $3.00 a package; demand for Meaties would still be at least 50,000 per month at that price." If we increase the price of Meaties by $0.20 to $3.00 per package, how does this change the optimal production plan and Healthy's profits?

The only thing that changes is the profit per package for Meaties, which increases from $0.65 to $0.85, so we want to

Maximize $$z = \$0.85\,M + \$0.45\,Y$$

Using the objective function sensitivity analysis information in Figure B-4 or re-solving the problem with the new objective function, we determine that the optimal production plan is still $M = 50,000$, $Y = 100,000$, but the month profit increases to $87,500. Thus with the available resources, the change in the relative profit margins for the two dog foods is not enough to change the optimal product mix, but the profit does increase. The same type of analysis would apply if we believe that the previously assumed material or packaging costs are not correct.

This analysis can be performed using a variety of scenarios for product and raw material prices. It shows us if, and how, the optimal product mix changes depending on the actual values of the parameters.

Pricing Additional Resources Returning to the original problem, suppose someone offered to sell additional cereal or meat, but at a higher price than Healthy is currently paying. The obvious questions are these:

1. What should Healthy be willing to pay for the additional resources?
2. How much of each resource should Healthy buy at various prices?
3. How should the additional resources be used?

We answer these questions using the dual prices discussed in the previous section. The dual price for the cereal constraint is $0.0625. This means that Healthy can buy another pound of cereal at the original price of $0.20 per pound and then, by revising its production could earn another $0.0625. Therefore, Healthy is willing to pay up to $0.20 + $0.0625 = $0.2625 per pound for additional cereal. Thus Healthy should buy at least one additional pound of cereal as long as its price is less than $0.2625 per pound. Should it buy more? The upper limit on B(1) in Figure B-5(c) is 600,000, so the answer is that Healthy should buy up to an additional 200,000 pounds of cereal (with no additional meat). Once it buys 200,000 additional pounds of cereal, its optimum production plan is to make 200,000 packages of Yummies and no Meaties. Additional cereal, even at $0.01 per pound, will not help Healthy unless it is able to obtain additional meat as well.

When only one resource is changed at a time, the dual price and right-hand-side ranging information for the original model can be used to determine what price to pay for an additional resource and how much to buy. The new optimum solution, however, is not immediately clear and requires some additional arithmetic. If two or more resources are changed simultaneously, the model usually needs to be changed and the problem re-solved. With existing computer packages, such modifying and resolving are easy.

Evaluating the Effects of Additional Constraints Optimization models are also used to evaluate the consequences of constraints. Consider the original Healthy Pet Food problem again. Suppose a shortage of transportation vehicles has developed, and Healthy's shipper has informed Healthy that it can only ship 135,000 packages of dog food each month (a 10% reduction from Healthy's current optimum level). What should Healthy do, and how much does this restriction cost Healthy each month?

A typical response is to reduce production of all products by 10%—that is, make 45,000 packages of Meaties and 90,000 packages of Yummies. This generates a monthly profit of $69,750, a 10% reduction, and is not the appropriate response. By adding the constraint $M + Y \leq 135,000$ to the original model and solving, we get the solution that $M = 65,000$, $Y = 70,000$, and $z = \$73,750$. Thus even though total production must be cut by 10%, it is best to increase the production of Meaties. The total profit decreases by less than 5%. This suggests that Healthy should only be willing to pay up to $3750 per month for additional monthly shipping capacity of 15,000 packages, not $7750. This result is counterintuitive for many decision makers, but we can see that the difference in profit between the optimum plan and the across-the-board reduction is substantial.

Incorporating Product Demand Functions into the Model Linear programs dealing with product mix and resource allocation decisions typically assume that prices are fixed by some outside force and that the company's decision will not influence these prices. In some cases, however, companies control the prices of their outputs and inputs through the amount they produce or purchase. For example, if the company lowers the price of its product, it believes that it can sell more. In these cases, the company is interested in determining its optimal production or resource allocation strategy and in determining the optimal price(s) for some product(s). When the underlying demand functions, which describe the relationship between product price and demand, exhibit the right properties, linear programming models are used to solve the price and volume problems simultaneously.

Suppose Healthy believes the amount of each dog food it can sell depends on the price it sets. To keep things simple, we assume that over reasonable ranges or production the demand functions for Meaties and Yummies can be described by Table B-12.

One way to solve the problem is to construct nine models, one for each combination of product prices. Each model has objective function coefficients based on the assumed product prices, and the upper limits on product demand vary with the price. After solving all nine problems, we find the optimum solution is to set the price of Meaties at $3.10 per package and the price of Yummies at $2.00 per package, producing 35,000 packages of Meaties and 100,000 packages of Yummies each month. This generates a monthly profit of $78,250. (Solve these problems yourself to verify the result.)

Table B-12 Healthy Pet Food Demand Function	
Meaties	
If price per package is	Market demand per month is
$3.10	35,000
$2.80	60,000
$2.60	100,000
Yummies	
If price per package is	Market demand per month is
$2.25	50,000
$2.00	100,000
$1.75	140,000

SUPPLEMENT HIGHLIGHTS

1 A mathematical model represents the essential features of an object, system, or problem without unimportant details. Analyzing and manipulating the model gives insight into how the real system behaves under various conditions. From this we determine the best system design or action to take.

2 Constrained optimization models have three major components: decision variables, objective function, and constraints. Decision variables are physical quantities controlled by the decision maker and represented by mathematical symbols. The objective function defines the criterion for evaluating the solution. The constraints are a set of functional equalities or inequalities that represent physical, economic, technological, legal, ethical, or other restrictions on what numerical values can be assigned to the decision variables.

3 The main benefit of optimization models is the ability to evaluate possible solutions in a quick, safe, and inexpensive way without actually constructing and experimenting with them. Other benefits include the following: they structure the thought process, increase objectivity, make complex problems more tractable, make problems amenable to mathematical and computer solution, and facilitate "what if" analysis.

4 Linear programs are constrained optimization models that satisfy three requirements. The decision variables must be continuous; they can take on any value within some restricted range. The objective function must be a linear function. The left-hand sides of the constraints must be linear functions.

5 Model formulation is the most important and the most difficult aspect of solving a real problem. Solving a model that does not accurately represent the real problem is useless. Model formulation includes identifying and defining the decision variables for the problem, defining the objective function, and identifying and expressing mathematically all of the relevant constraints.

6 The characteristic that makes linear programs easy to solve is their simple geometric structure. A solution for a linear program is any set of numerical values for the variables. A feasible solution is a solution that satisfies all of the constraints. The feasible set or feasible region is the set of all feasible solutions. Finally, an optimal solution is the feasible solution that produces the best objective function value possible.

7 We can graphically solve an LP problem by using the fundamental theorem of linear programming: If a finite optimal solution exists, then at least one extreme point is optimal. To find the exact coordinate values for the optimum from the graphs, we identify the constraints that intersect to form the extreme point. We solve simultaneously the equations corresponding to the constraints to find the point that lies on both lines (the extreme point).

8 The simplex algorithm has many desirable features in practice. It only moves from one extreme point to a better or equally good extreme point, thereby skipping large numbers of suboptimal extreme points without explicitly identifying them. When it finds an optimum, it identifies this fact and stops. The algorithm detects whether the problem is infeasible, is unbounded, or has multiple optima. The algorithm uses very simple mathematics that are easy to implement on a computer.

9 In the Big-M method, we make the objective function coefficients for the artificial variables either $-$infinity for maximization problems or $+$infinity for minimization problems. The name comes from the fact that the letter M represents some large number that replaces infinity. This method encourages the algorithm to drive the artificial variables to zero by kicking them out of the basis.

10 Most packages offer several output options: to display the optimum, the optimum with sensitivity analysis, the initial simplex tableau and the optimum, and so forth.

11 The formulations earlier in the supplement give the impression that using linear programming is a clean, simple process. In practice, using linear programming and other optimization models is not so straightforward, nor is it static. Specifically, our goal in using models is to obtain usable solutions that are better than those we would have obtained without the models, to use the models to revise and update our decisions in a timely fashion, and to increase our confidence in our decisions.

KEY TERMS

model B2
constrained optimization model B2
decision variables B2
objective function B2
constraints B2
linear program B5
sensitivity analysis B6
parameters B6
coefficients B6
proportionality B7
additivity B7
divisibility B7

certainty B7
feasible solution B14
feasible region B14
feasible set B14
optimal solution B14
extreme points B16
simplex method B17
simplex algorithm B17
canonical form B18
nonbasic variables B20
basic variables B20
basic solution B20

simplex tableau B21
pivot column B23
ratio test B23
pivot row B23
pivot element B23
artificial variables B26
Big-M method B27
two-phase method B28
degenerate solution B28
dual price B31
shadow price B31

SOLVED PROBLEMS

■ Solved Problem 1

Suppose a company manufactures two products, A and B, using three inputs, labor, material R, and materials S. To make one unit of product A requires 6 pounds of R, 7.5 pounds of S, and 9 person-hours of labor; to make one unit of product B requires 12 pounds of R, 4.5 pounds of S, and 6 person-hours of labor. The demands for the products are such that the company can sell as much of each product as it can produce and earn a profit of \$3 per unit of A and \$4 per unit of B. However, only 900 pounds of R, 675 pounds of S, and 1200 person-hours of labor are available to the company each day.

a. Formulate the company's problem as a linear program to maximize profit.
b. Graph the feasible region for this problem.
c. Solve the problem graphically by finding the best extreme point.

Solution:

a. The decision maker controls the amount of each product to make each day. Thus we can define the decision variables as

x_j = No. of units of product j $(= A, B)$ to make each day.

The company's objective is to maximize the profit per day. We want to express "profit per day" as a function of the decision variables so that no matter what numerical values the decision variables take on, the function will compute the daily profit. Total profit per day is equal to

(profit/unit of A made) $\times$ (units of A made/day)
  $+$ (profit/unit of B made) $\times$ (units of B made/day)

Substituting the appropriate numerical values and variables for these quantities gives an objective function:

Maximize $z = 3x_A + 4x_B$

The firm wants to maximize this objective function but subject to satisfying constraints on the availability of the inputs. The first restriction is that

No. of person-hours used per day $\leq$ 1200

The number of person-hours used per day is equal to

(person-hours used/unit of A) $\times$ (units of A made/day)
  $+$ (person-hours used/unit of $B)$ $\times$ (units of B made/day)

Substituting in the appropriate coefficients and variable names yields the constraint

$$9x_A + 6x_B \leq 1200$$

Repeating this for the raw material restrictions yields the constraints

$$6x_A + 12x_B \leq 900$$
$$7.5x_A + 4.5x_B \leq 675$$

Finally the variables must be nonnegative:

$$x_A, x_B \geq 0$$

Putting all of this together gives the linear programming model:

Maximize $z = 3x_A + 4x_B$

Subject to $9x_A + 6x_B \leq 1200$
                      $6x_A + 12x_B \leq 900$
                      $7.5x_A + 4.5x_B \leq 675$
                      $x_A, x_B \geq 0$

b. The nonnegativity constraints, $x_A \geq 0$ and $x_B \geq 0$, restrict the feasible solutions to the upper right-hand quadrant of Figure B-6. We then plot the inequality $9x_A + 6x_B \leq$

Figure B-5

Graphical solution of Solved Problem B-1
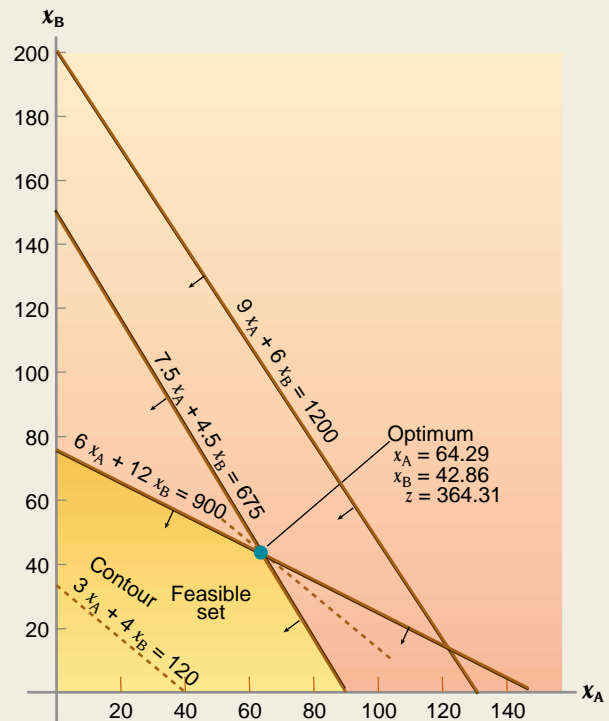
1200 by graphing its boundary line, $9x_A + 6x_B = 1200$, and finding the side that satisfies the constraint. We first set $x_A = 0$ and solve for x_B, which gives $x_B = 200$, and then set $x_B = 0$ and solve for x_A, which gives $x_A = 133.33$. The point $x_A = 0$, $x_B = 0$ satisfies the inequality, so all the points on the same side of the line satisfy this inequality. Repeating this procedure for the other constraints gives the feasible set in Figure B-5. Note that the first constraint is redundant; that is , it does not eliminate any solutions that are not already eliminated by the other constraints. It can be eliminated from the problem without changing the feasible set. The feasible set is the four-sided shaded region.

c. Figure B-5 shows objective function contours graphed for various profit levels (dotted lines). The contour with the highest value intersects the extreme point formed by the intersection of the second and third constraints. To obtain the coordinates of this point, we solve the two constraint equations simultaneously:

$$\left. \begin{array}{l} 6x_A + 12x_B = 900 \\[8pt] 7.5x_A + 4.5x_B = 675 \end{array} \right\} \rightarrow x_A = 64.29 \quad x_B = 42.86$$

The objective function value for the highest contour is then $z = 3(64.29) + 4(42.86) = 364.31$.

■ Solved Problem 2

Chip Green is the head groundskeeper at Birdie Valley Golf Club. For the mix of grass for the golf course, Chip has decided that the best fertilizer would be a 10-8-12 mixture. (Fertilizer is defined by three values—a, b and c—where a is the percentage of nitrogen, b is the percentage of phosphorus, and c is the percentage of potash in the fertilizer. The remaining material is inert matter.) Chip can buy a 10-8-12 mix of fertilizer for $21.75 per 100 pounds, but there are other fertilizers on the market at a variety of prices. The chemical content and prices are given below. Chip would like to determine whether or not he could buy several fertilizers and mix them together to obtain a 10-8-12 mixture at a lower cost than $21.75 per 100 pounds. Recognizing that it might be impossible to obtain an exact 10-8-12 mix from the fertilizers, Chip is willing to accept chemical percentages of at least the target amounts, but no more than 0.5% above them (so the nitrogen level should be between 10% and 10.5%).

a. Formulate Chip's problem as a linear program.
b. Solve this problem using a computer package.

Fertilizer	%Ni %Ph %Po	Cost/100 lb
1	10-8-12	$21.75
2	8-11-15	$23.50
3	12-7-12	$22.00
4	10-10-10	$19.50
5	15-10-6	$18.50

Solution:

a. Chip is interested in the mix of fertilizers to use, so we can define the variables

$$x_j = \text{lb of fertilizer } j \text{ in each lb of mixture}$$

The objective is to minimize the cost per pound of fertilizer:

$$\text{Minimize } z = 0.2175\, x_1 + 0.2350\, x_2 + 0.2200\, x_3 + 0.1950\, x_4 + 0.1850\, x_5$$

The main constraints are those that ensure the desired percentage of nitrogen, phosphorus, and potash.

$$
\begin{aligned}
10x_1 + 8x_2 + 12x_3 + 10x_4 + 15x_5 &\geq 10 \\
10x_1 + 8x_2 + 12x_3 + 10x_4 + 15x_5 &\leq 10.5 \\
8x_1 + 11x_2 + 7x_3 + 10x_4 + 10x_5 &\geq 8 \\
8x_1 + 11x_2 + 7x_3 + 10x_4 + 10x_5 &\leq 8.5 \\
12x_1 + 15x_2 + 12x_3 + 10x_4 + 6x_5 &\geq 12 \\
12x_1 + 15x_2 + 12x_3 + 10x_4 + 6x_5 &\leq 12.5
\end{aligned}
$$

Because the variables are defined as the amount of each type of fertilizer in 1 pound of mixture, we need a constraint that ensures that the mixture adds up to exactly 1 pound:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1$$

Also, all variables must be nonnegative.

b. The optimal solution for this problem is $x_1 = 0.225$, $x_2 = 0.15$, $x_3 = 0.40$, $x_4 = 0.225$, $x_5 = 0$, and $z = \$0.21605$. By mixing the 10-8-12 fertilizer with three other fertilizers, Chip can reduce the cost. The actual mixture has a chemical content of 10.5-8.5-12.

■ Solved Problem 3

Solve the following linear program using the simplex method.

Minimize $z = 30x_1 + 10x_2$

Subject to
$$2x_1 + 4x_2 \leq 80$$
$$x_1 + x_2 \leq 25$$
$$8x_1 + 6x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

Solution:

We first must put the problem in canonical form. All the right-hand sides are nonnegative. To change the constraints into equalities, we (1) add a slack variable s_1 to the first constraint, (2) add a slack variable s_2 to the second constraint, and (3) *subtract* a slack variable s_3 from the third constraint. Variable s_1 and s_2 will be the initial basic variables for constraints 1 and 2, but constraint 3 does not have a basic variable (one that appears only in constraint 3 and with a +1 coefficient), so we add an artificial variable, A_3, to constraint 3.

We now construct the tableau shown in Table B-13. Because we are minimizing, we assign the artificial variable A_3 a large positive objective function coefficient—say, +1000. For the initial solution we compute the z_j and $c_j - z_j$ values for each column. Because we are minimizing the objective function, we select the variable with the most *negative* $c_j - z_j$ value to enter the basis; this is variable x_1. We now perform the ratio test by dividing each of the right-hand-side constants by the corresponding values in the pivot column (the x_1 column). The row with the smallest ratio is the one for constraint 3, so x_1 will enter the basis and A_3 will leave the basis.

We now perform the pivoting operations. We divide constraint 3 by the pivot element, 8. We take 2 times the new form of constraint 3 and subtract it from constraint 1 to give a zero value in the pivot column of constraint 1. Similarly, we subtract 1 times the new form of constraint 3 from constraint 2. The resulting tableau is given in Table B-14. In this table the $c_j - z_j$ value for x_2 is most negative (it

Table B-13 First Tableau: Solved Problem 3

C_B	Basic Variable	x_1 30	x_2 10	s 0	s_2 0	s_3 0	A_3 1,000	i	Ratio	
0	s_1	2	4	1	0	0	0	80	(80/2) = 40	
0	s_2	1	1	0	1	0	0	25	(25/1) = 25	
1,000	A_3	8	6	0	0	−1	1	120	(120/8) = 15	←Pivot row
	z_j	8,000	6,000	0	0	−1,000	1,000	120,000		
	$c_j - z_j$	−7,970	−5,990	0	0	1,000	0			

<div align="center">↑
Pivot
column</div>

Table B-14 Second Tableau: Solved Problem 3

C_B	Basic Variable	x_1 30	x_2 10	s_1 0	s_2 0	s_3 0	A_3 1,000	b	Ratio	
0	s_1	0	5/2	1	0	1/2	—	50	(50/2.5) = 20	←Pivot row
0	s_2	0	1/4	0	1	1/8	—	10	(10/0.25) = 40	
30	x_1	1	3/4	0	0	−1/8	—	15	(15/0.75) = 20	
	z_j	30	90/4	0	0	−30/8	—	450		
	$c_j - z_j$	0	−50/4	0	0	30/8	—			

<div align="center">↑
Pivot
column</div>

is the only one that is negative), so we bring x_2 into the basis. Constraints 1 and 3 tie for the lowest ratio, so we arbitrarily select constraint 1 as the pivot row (actually, there are tie-breaking rules, but they are beyond the scope of this book). Performing the pivot operations gives the tableau in Table B-15. No $c_j - z_j$ value is positive in this table, so the solution there is optimal. To get the solution, we set the nonbasic variables, s_1 and s_3, equal to zero and the basic variables equal to the values in the "b" column: $x_1 = 0$, $x_2 = 20$, and $s_2 = 5$, and the objective function value is $z = 200$. (Notice that this is a degenerate solution because a basic variable equals zero.)

Table B-15   Third Tableau: Solved Problem 3

C_B	Basic Variable	x_1 30	x_2 10	s_1 0	s_2 0	s_3 0	A_3 1,000	b
10	x_2	0	1	2/5	0	1/10	—	20
0	s_2	0	0	−1/10	1	1/10	—	5
30	x_1	1	0	−3/10	0	−2/10	—	0
	z_j	30	10	25	0	25	—	200
	$c_j - z_j$	0	0	5	0	5	—	

DISCUSSION QUESTIONS

1. What are the three primary components of a constrained optimization model?
2. Explain the difference between a parameter and a decision variable.
3. What are the primary assumptions underlying linear programming models?
4. If most parameters represent estimates that are not known with certainty, how can constrained optimization models be of any value for decision making?
5. What does it mean when a problem has an unbounded solution? If you formulated a real problem and the solution? If you formulated a real problem and the solution was unbounded, what does this imply about your model?
6. What does it mean to perform sensitivity or "what if" analysis?
7. Explain the meaning of dual prices. Why would one want to know the value of a dual price, and how might it be used?

PROBLEMS

1. Deutchlander Machine Company (DMC) makes two types of printing presses: a four-color litho and a two-color litho. Demand for both products is booming and exceeds DMC's ability to satisfy it. Most raw materials are plentiful, but production is limited by three factors: a shortage of steel rollers, gear cutting capacity, and roller polishing capacity. Each four-color press requires 16 rollers, and each two-color press requires 8 rollers. The four-color presses require 30 hours of gear cutting and 8 hours of polishing time, and the two-color presses require 12 hours of gear cutting and 3 hours of polishing time. DMC is able to buy 100 rollers per week, and it has 160 hours of gear cutting time and 40 hours of polishing time available per week.

To avoid harming long-term sales, DMC does not want to raise prices. At current prices DMC will earn a profit of DM24,000 on each four-color press made and DM10,000 on each two-color press made. So as not to abandon either product market, DMC also wants to produce at least two units of each press each week.

a. Formulate DMC's problem as a linear program to maximize its profit.

b. Solve the problem (either graphically, using the simplex method, or using a computer) and explain in words DMC's optimal solution.

2. Robertville Furniture Company (RFC) makes two types of dressers: the Classic and the Modern. RFC sells the Classic for $310 and the Modern for $350. At these prices RFC believes it can sell up to 20 Classic and 25 Modern dressers per week. Each type of dresser is made of oak and covered with a pecan veneer. The wood requirements for each dresser are given in the table. Oak costs $1.50 per board foot, and pecan veneer costs $2 per square foot. The dressers must go through two manufacturing departments: cutting and veneering. The machine hours required for each dresser are also given below. RFC has only 40 hours of production

	Oak/ Dresser (board ft)	Pecan/ Dresser (sq/ft)	Cutting Time (hr/dresser)	Veneering Time (hr/dresser)
Classic	90	50	1.20	1.00
Modern	100	60	0.90	1.20

time available each week in each department, and it can obtain up to 2000 board feet of oak and 1500 square feet of pecan veneer each week. The cost of operating each manufacturing department is essentially fixed, so RFC would like to maximize its contribution to overhead (revenue minus variable costs).

a. Formulate RFC's problem as a linear program, defining your variables clearly.

b. Solve the problem (graphically, using the simplex method, or using a computer) and state what RFC should do (i.e., explain RFC's optimal solution in words).

c. What resource is limiting RFC's production?

3. Western Pulp (WP) produces recycled paperboard for box manufacturers by combining four grades of recycled paper stock. Each grade of stock has a different strength, color, and texture. The strength, color, and texture of the paperboard are approximately a weighted average of those characteristics of the paper inputs. The table gives the characteristics of the paper stocks and their cost per ton. WP has received an order for 500 tons of paperboard with a strength rating of at least 7, a color of at least 5, and texture of at least 6. WP would like to determine the least costly mix required to produce this paperboard.

Paper Grade	Strength	Color	Texture	Cost/Ton
1	8	9	8	$150
2	6	7	5	$110
3	5	5	6	$ 90
4	3	4	5	$ 50

a. Formulate the problem as a linear program.

b. Solve this problem on a computer and explain the optimal solution.

4. Volcano Potato Company (VPC) grows potatoes, processes them, and then sells three potato products: fresh potatoes, frozen french fried potatoes, and frozen hash ball potatoes (shredded and then reconstituted balls of potatoes with a soft consistency). During the next two months, VPC expects to harvest 8 million pounds of potatoes. VPC would like to determine how much of each product should be made from the potatoes. Potatoes are graded according to quality on a 0–5 scale. VPC divides its potatoes into three grades: A, B, and C. Grade A potatoes have an average quality rating of 4.5; grade B potatoes have an average quality rating of 2.5, and grade C potatoes have a quality rating below 1 and are not used for any products. From historical data and estimates based on the current growing season's weather, VPC believes the distribution of potato quality will be:

Grade	A	B	C
% of Harvest	50	40	10

Fresh potatoes earn a profit of $0.40 per pound after processing costs, but only grade A potatoes can be sold as fresh potatoes. Frozen french fried potatoes earn $0.32 per pound after processing costs, but the potatoes used must have an average quality rating of at least 3.5. Hash balls earn $0.25 per pound after processing costs, but the potatoes used must have an average quality rating of at least 3.0. Assume that these ratings are linear in the sense that the quality rating of a mixture equals the weighted average of the inputs. VPC believes it can sell as much french fried and hash ball potatoes as it can make, but it believes the total demand for its fresh potatoes during the next two months is 2.5 million pounds.

a. Formulate a linear programming model to determine the best use for the potatoes so as to maximize VPC's profit.

b. Solve the problem using a computer and explain the answer in words.

c. Suppose VPC could buy additional grade A potatoes at $0.35 per pound; how much should it buy? Explain.

5. Manfred Leaks manages a large discount store. His biggest problem has been scheduling cashiers so that he has an adequate number without having too many. The store is open from 9 A.M. to 9 P.M. every day of the week. Based on historical data, he found that the customer patterns for Monday to Thursday are essentially the same, but those for Friday, Saturday, and Sunday are all different. He divided the day into three 4-hour segments and estimated how many cashiers were needed for each time period for each day of the week. These are given in the following table.

Day of Week	Mon–Thur	Fri	Sat	Sun
9 A.M.–1 P.M.	6	3	10	4
1 P.M.–5 P.M.	5	8	14	12
5 P.M.–9 A.M.	8	4	7	6

Employees must work continuous 8-hour shifts beginning at 9 A.M. or 1 P.M., and their weekly schedules must be made up of 5 consecutive days of work with 2 consecutive days off (and they work the same hours each workday). Manfred would like to devise weekly schedules that will minimize the total number of cashiers needed, but the schedules must be such that the minimum cashier requirements in the table are satisfied.

a. Formulate Manfred's problem as a linear program; be sure to define the variables precisely. (*Hint*: There are 14 possible schedules; there will be one variable corresponding to each schedule.)

b. Solve the problem using a computer.

c. Is the assumption of divisibility satisfied? Will your answer to this question be true in general? Explain.

6. Checker Credit Service provides credit information to its customers throughout the country 24 hours a day. Rosalind Hanks is the manager of phone services. She supervises credit reps who answer customers' calls. From historical data she has estimated that the following number of credit reps are needed during various times of the day.

Time Period	No. of Reps Needed
Midnight–4 A.M.	3
4 A.M.–8 A.M.	6
8 A.M.–noon	13
Noon–4 P.M.	15
4 P.M.–8 P.M.	12
8 P.M.–midnight	9

Employees work shifts of 8 consecutive hours, and shifts can start at the beginning of any of the six periods shown in the table. Ms. Hanks has complete freedom in deciding the *number* of days each employee works each week, so she is interested only in knowing how many employees should start work at the beginning of each time period to minimize the total number of employees needed each day.

a. Formulate her problem as a linear program. (*Hint*: There will be six decision variables.)

b. Solve the problem using a computer.

c. Is the divisibility assumption satisfied?

d. Suppose full-time employees were paid $8 per hour, and suppose part-time employees could be hired to work 4-hour shifts for $5 per hour. But part-time employees are only half as efficient as full-time employees are only half as efficient as full-time employees (i.e., Checker needs two part-time employees to do the work of one full-time employee). Formulate and solve the new problem.

7. Ronald Stimpson is regional sales manager for an industrial products company. His region contains four sales districts. He supervises 10 sales representatives and controls an advertising budget of $800,000 per year. Mr. Stimpson believes that sales in a district are related to the amount of advertising and the number of sales reps assigned to the district. Based on historical data, his estimates of incremental annual sales per dollar of advertising and sales per sales rep for each district are given in the following table. Mr. Stimpson would like to determine how to allocate sales reps and the advertising budget among the four districts to maximize

District	1	2	3	4
$ sales/$ advertising	7	12	10	11
$ sales (0000)/sales rep	38	25	36	41

sales. To maintain a reasonable presence in each market, however, each district must be assigned at least one sales rep, and at least $50,000 must be spent on advertising in each market.

a. Formulate a linear programming model to help Mr. Stimpson solve his problem.

b. Solve the problem using a computer.

8. Great Plains Advertising Company (GPAC) has a $150,000 advertising budget to advertise an automobile firm. GPAC is considering advertising in newspapers and on television. The more GPAC advertises in a particular medium, the less effective additional ads are in reaching new customers. The following table lists the number of new cus-

	No. of Ads	New Customers/Ad
Newspaper	1–10	900
	11–20	600
	21–30	300
Television	1–5	10,000
	6–10	5,000
	11–15	2,000

tomers reached by each ad. Each newspaper ad costs $1500, and each television ad costs $10,000. At most 30 newspaper ads and 15 television ads can be placed. Also, GPAC would like to use at least as many newspaper ads as television ads. Formulate a linear program that will maximize the number of new customers contacted subject to the previous restrictions. (*Hint:* You need six variables, not two.)

9. The demand for a company's product during the next four months is given here. It costs the company $9 per unit

Month	Demand
1	450
2	575
3	490
4	530

to produce the product using regular-time labor and $12 per unit to produce the product using overtime labor. A maximum of 500 units can be made each month using regular-time labor. There is no limit on overtime productin. Excess units produced can be stored at a costs of $2 per unit per month.

a. Formulate this company's problem as a linear program to minimize its total production and storage costs during the next four months and to satisfy demand (there are no initial inventories).

b. Solve the problem using a computer.

10. Tavisbond Manufacturing Company makes high-grade pipe for the oil and chemical industries. Tavisbond must plan its production for the next 7 months, March to September. The forecast demands (in thousands of feet) for its pipe are as follows:

Mar	Apr	May	June	July	Aug	Sept
40	60	70	80	90	100	80

Tavisbond can make 75,000 feet of pipe per month using regular-time production at a cost of $1.25 per foot. Tavisbond can make an additional 15,000 feet of pipe each month using overtime production at a cost of $1.50 per foot. Any pipe made in one month and sold in a subsequent month incurs an inventory holding cost of $0.15 per foot per month. Tavisbond expects to end February with 5000 feet of pipe in inventory and would like to end September with 10,000 feet of pipe in inventory.

   a. Formulate Tavisbond's problem as a linear program to minimize its total cost during the next 7 months and to ensure that it delivers the predicted amounts of pipe on time.

   b. Solve the problem using a computer.

11. Metallica Manufacturing Company has seen the demand for two new types of metal alloys explode in recent months. The alloys, which are made by mixing copper, nickel, and aluminum, do not require an exact formulation of components but must satisfy the following general specifications: Copper costs $2 per pound, nickel $3 per pound, and aluminum $1.50 per pound. Metallica has a limit of 2000 pounds of copper, 3000 pounds of nickel, and 4000 pounds of aluminum available each day. Assume that the company can sell as much of each alloy as it makes at a price of $5 per pound for alloy A and $6 per pound for alloy B.

Alloy A	Alloy B
At least 40% copper	No more than 35% copper
At least 10% nickel	At least 40% nickel
No more than 25% nickel	
Aluminum content must be exactly twice that of nickel	No more than 30% aluminum

   a. Formulate the company's problem as a linear program to maximize profit.

   b. Solve the problem using a computer.

12. Wilson Creek Farm has 200 acres of land available for planting. The owner is considering planting three crops: corn, soybeans, and wheat. The production yield, water requirements, and labor requirements for a salable crop are given here. The owner expects to have only 35,000 gallons of water available per week to use for the crops, and during the

Crop	Bushels/ Acre Produced	Water Required (gal/acre/week)	Person-Hours Labor Required/Acre
Corn	300	200	35
Soybeans	200	150	40
Wheat	80	125	30

growing season he will only have 8000 person-hours of labor available. The expected profit per bushel of each crop is $1.00 for corn, $1.60 for soybeans, and $3.00 for wheat. The owner can use any mix of crops (i.e., he can plant the same crop on all 200 acres or he can plant all three crops in different proportions).

   a. Formulate the problem as a linear program to find the profit-maximizing planting strategy.

   b. Solve the problem using a computer.

13. A manufacturing company has three manufacturing plants located throughout the country. The company supplies four major wholesalers distributed throughout the country with a product made at these three plants. The manufacturing cost is $12 per unit at plant A, $10 per unit at plant B, and $11 per unit at plant C. The shipping cost between each plant and each wholesaler is given here. The units

	Mfg. Plant	Wholesaler				Capacity
		1	2	3	4	
Transport	A	7	10	16	5	300
cost per unit	B	10	4	8	12	400
shipped	C	11	8	10	9	500
	Demand	150	300	350	250	

demanded by each wholesaler are also listed, along with the production capacity at each manufacturing plant. The company wishes to satisfy the demands of each wholesaler at minimum total cost (production plus shipping). However, no plant should be operating at less than 60% of its capacity.

   a. Formulate a linear program to solve the company's problem.

   b. Solve the problem using a computer.

14. AMCHEM Chemical Company produces three products: A, B, and C. Each product requires labor to produce it, and production of each product creates pollutants. By law the firm is not allowed to produce more than the following pollutants per day: 200 pounds of sulfur dioxide, 300 pounds of carbon monoxide, 150 pounds of hydrogen sulfide, and 50 pounds of benzene. The total number of person-hours of labor available per day is 6000. In addition, the total output per day of products A and B combined cannot be more than the output of product C. Each pound of product A generates

a profit of $5, each pound of B generates $7, and each pound of C generates $4. Pollutant and labor rates per hundred pounds of product are given here.

		A	B	C
Pounds of	Sulf dioxide	.10	.05	.20
Pollutants/	Carb monoxide	.18	.04	.03
100 lb	Hydrogen sulfide	.25	.15	.02
of Product	Benzene	.01	.06	.04
Pers-Hr/				
100 l.b of product	Labor	3.0	2.0	4.5

a. Formulate this problem as a linear program to maximize daily profit.

b. Solve the problem using a computer.

15. International Fiber Company (IFC) makes newsprint for newspapers. IFC has three paper mills, A, B, and C, spread throughout the country. The cost of producing newsprint varies from mill to mill. IFC estimates that the marginal production cost for each ton of newsprint is $210 at mill A, $225 at B, and $220 at C. IFC supplies five primary geographical markets from these three mills. The monthly demand at each market, the per ton shipping cost between each mill and each market, and the monthly production capacity of each mill are as follows:

	Mill	Market					Capacity (tons/ month)
		1	2	3	4	5	
Shipping	A	20	25	30	15	35	1200
Cost/Ton	B	30	20	32	28	19	1500
	C	25	18	28	23	31	900
Monthly demand		600	1000	500	800	500	

IFC would like to satisfy all market demands at minimum total cost (production plus transportation). Formulate a linear program to sove IFC's problem. (Hint: Let x_{ij} = tons of newsprint made at mill i and sent to market j each month, a total of 15 variables.)

16. Enviroclean Waste Disposal Company collects toxic wastes from commercial and industrial sites and transports them to one of its three treatment centers, where the materials are either incinerated, chemically treated to reduce their toxicity, or stored in an approved manner. Each treatment center has a limited capacity: centers A and B can process 50 tons per day, and center C can process 75 tons per day. Enviroclean serves six general districts. The round-trip travel and collection times from the six regions to the treatment centers and the predicted daily volumes of waste generated in each district are given in the following table.

District	Treatment Center*			Daily Volume (tons/day)
	A	B	C	
1	1.2	1.1	3.2	20
2	0.5	1.3	2.5	30
3	1.0	0.4	1.9	45
4	1.5	1.6	1.4	25
5	2.4	1.5	1.6	20
6	2.6	1.9	0.5	25

*Round-trip travel plus collection times in hours.

Each district is served by several trucks because each truck can handle approximately 5 tons of wastes per load. Trucks are stationed at each treatment center and then they go to a district to collect wastes and return to the center. The cost of collection is approximately proportional to the travel plus collection time because drivers are paid on a per-hour basis and do not work 8-hr days.

a. Enviroclean would like to determine how much of each district's wastes should be collected and sent to each treatment center to minimize collection cost. Formulate this problem as a linear program.

b. Solve the problem using a computer.

c. Does the fact that each truck can carry 5 tons of waste violate the divisibility assumption? Does it make the formulation and solution in (a) and (b) invalid? Explain.

17. The demand for Emca Inc.'s product during the next 4 months is given here. The product is made up of two compo-

Month	Demand
1	90
2	85
3	95
4	100

nents, A and B. Emca can either manufacture the components itself or subcontract to another company. If Emca manufactures the components, each component must go through two production departments: fabricating and finishing. Component A requires 2 hours of fabricating time and 4 hours of finishing time; component B requires 3 hours of fabricating time and 2 hours of finishing time. Each department has 550 hours of time available each month for production. Emca has determined that is costs $200 per unit of A that is made and $240 per unit of B. Emca believes that its production capacity is insufficient to satisfy the demand, so it has arranged to buy some units of A and B from a subcontractor for $220 per unit of A and $255 per unit of B. Any units of the components can be held in inventory at a cost of $8 per month.

a. Formulate Emca's planning problem as a linear program to minimize its cost during the next 4 months while delivering the amount demanded each month.

b. Solve the problem using a computer.

18. (The previous problems all dealt with the issues faced by private firms. The following is a simplified version of an actual public sector problem.) The Appleville School District has two high schools, each of which has a capacity of 4000 students. Approximately one-third of the high school students in the district are members of racial minorities. The district can be divided into four distinct communities. The number of students in each community expected to attend a public high school next year and the distance from the center of each community to each high school are listed in Table B-16.

Table B-16 Appleville School District Data

Community	No. of Majority Students	No. of Minority Students	Miles to H.S A	Miles to H.S B
1	1900	250	3.4	1.5
2	1700	400	2.4	2.2
3	800	650	1.1	2.9
4	550	1250	1.7	2.8
Total	4950	2550		

Historically the two high schools have been racially un-balanced, with school A having a disproportionately high enrollment of minority students and school B having a dispro-portionately high enrollment of majority students. To satisfy a court agreement to achieve better racial balance between the schools, each high school must have at least 24% and no more than 44% of its enrollment made up of minority students (the overall minority enrollment in the district is 34%). The school district would like to determine how many students of each type (majority and minority) should be sent from each community to each high school to minimize total student bus miles traveled.

a. Formulate this problem as a linear program. (*Hint*: let

x_{ij} = number of majority students from community i assigned to H.S. j

y_{ij} = number of minority students from community i assigned to H.S. j

where i = 1, 2, 3, 4 and j = A, B.)

b. Solve this problem using a computer.

19. Solve the following linear program graphically:

Minimize $z = 10A + 7.5B$
Subject to $\quad 4A + \quad 2B \geq 10$
$\quad -3A + \quad 2B \leq 3$
$\quad A + \quad B \leq 3$
$\quad A, B \geq 3$

20. Solve the following linear program graphically.

Maximize $\quad z = x_1 + 2x_2$
Subject to $\quad 6x_1 + 3x_2 \leq 15$
$\quad 2x_1 - \quad x_2 \geq 4$
$\quad x_1, x_2 \geq \quad 0$

21. Solve the following linear program graphically.

Minimize $\quad z = 4A + \quad B$
Subject to $\quad 3A + 2B \geq 12$
$\quad 2A - 6B \leq -18$
$\quad A - \quad B \leq 2$
$\quad A, B \geq 0$

22. Solve the following linear program using the simplex algorithm.

Maximize $\quad z = 2A + 4B + 3C$
Subject to $\quad 3A + 4B + 2C \leq 60$
$\quad 2A + \quad B + 2C \leq 40$
$\quad A + 3B + 2C \leq 80$
$\quad A, B, C \geq 0$

23. Solve the following linear program using the simplex algorithm.

Maximize $\quad z = 5.0X + 8.0Y$
Subject to $\quad 2.5X + 5.0Y \leq 50$
$\quad 2.0X + 1.0Y \leq 20$
$\quad 7.5X + 2.0Y \leq 60$
$\quad X, Y \geq 0$

24. Solve the following linear program using the simplex algorithm.

Maximize $\quad z = 15x_1 + 25x_2 + 10x_3$
Subject to $\quad -6x_2 + 3x_3 \geq 12$
$\quad x_1 + 4x_2 + 2x_3 = 10$
$\quad \text{All } x_j s \geq 0$

25. Solve the following linear program using the simplex algorithm.

Minimize $\quad z = 3A + 4B + \quad C$
Subject to $\quad 2A + \quad B + \quad C \geq 20$
$\quad A + \quad B + 5C = 15$
$\quad A, B, C \geq 0$

26. Solve the following linear program using the simplex algorithm.

Maximize $\quad z = 4x_1 + 2x_2 - 2x_3$
Subject to $\quad 3x_1 + 6x_2 - 3x_3 \leq \quad 90$
$\quad 3x_1 + \quad x_2 + \quad x_3 \leq 180$
$\quad x_1 - \quad x_2 + \quad x_3 = \quad 60$
$\quad \text{All } x_j s \geq 0$

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