Revision of estimation

Example

Random variable y has geometrical distribution

$$f(y) = p(1-p)^y, \quad p \in (0,1), \, y \in \{0,1\}.$$

We have a sample $Y = \{y_i\}_{i=1}^t$ from this distribution. Derive

1. likelihood,
2. ML estimate of the parameter p
3. posterior pdf.

Solution

1. Product of two models:

$$p(1-p)^{y_1} p(1-p)^{y_2} = p^2 (1-p)^{y_1+y_2}$$

Likelihood (product of t models)

$$L(p) = p^t (1-p)^{\sum_{i=1}^t y_t} = p^{\kappa_t} (1-p)^{S_t}$$

where $\kappa_t = t$, $S_t = \sum_{i=1}^t y_i$ which are statistics of estimation.
The evolution of statistics is

$$\begin{aligned} S_\tau &= S_{\tau-1} + y_\tau \\ \kappa_\tau &= \kappa_{\tau-1} + 1 \end{aligned}$$

with the initial $S_0 = 0$, $\kappa_0 = 0$.

2. ML estimate - at the maximum of likelihood

$$\frac{d}{dp} L(p) = \frac{d}{dp} p^{\kappa_t} (1-p)^{S_t} =$$

$$= \kappa_t p^{\kappa_t - 1} (1-p)^{S_t} - p^{\kappa_t} S_t (1-p)^{S_t - 1} = 0$$

$$\kappa_t = p(S_t + \kappa_t) \quad \rightarrow \quad \hat{p}_t = \frac{\kappa_t}{S_t + \kappa_t}$$

3. Posterior pdf - is a product of likelihood and the prior pdf

$$f(p|Y) = p^{\kappa_t} (1-p)^{S_t} f(p).$$

Now, $f(p)$ can be selected in the form of likelihood with prior data. It is

$$f(p|Y, prior) = p^{\kappa_t} (1-p)^{S_t} p^{\kappa_0} (1-p)^{S_0} = p^{\kappa_t + \kappa_0} (1-p)^{S_t + S_0} =$$

$$= p^{\tilde{\kappa}_t} (1-p)^{\tilde{S}_t}$$

with new statistics $\tilde{\kappa}$ and $\tilde{S}$ with the same evolution but different initialization

$$\tilde{S}_\tau = \tilde{S}_{\tau-1} + y_\tau$$
$$\tilde{\kappa}_\tau = \tilde{\kappa}_{\tau-1} + 1$$

$\tilde{S}_0 = S_0$, $\tilde{\kappa}_0 = \kappa_0$.

Now, it holds: κ_0 is number of initial data and S_0 is sum of initial data.

If our prior belief is that the average of Y is $\bar{y}$, we can choose S_0 and κ_0 so that it holds

$$\bar{y} = \frac{S_0}{\kappa_0}.$$

The common size of S_0 and κ_0 expresses the importance if the prior belief. If we are sure about our guess, we choose them big. If not, they will be small.