Curvatures of Smooth and Discrete Surfaces

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The curvatures of a smooth curve or surface are local measures of its shape. Here we consider analogous quantities for discrete curves and surfaces, meaning polygonal curves and triangulated polyhedral surfaces. We find that the most useful analogs are those which preserve integral relations for curvature, like the Gauß/Bonnet theorem. For simplicity, we usually restrict our attention to curves and surfaces in euclidean three space $\mathbb{E}^3$, although many of the results would easily generalize to other ambient manifolds of arbitrary dimension.

Most of the material here is not new; some is even quite old. Although some references are given, no attempt has been made to give a comprehensive bibliography or an accurate picture of the history of the ideas.

1. Smooth curves, Framings and Integral Curvature Relations

A companion article [Sul06] investigated the class of FTC (finite total curvature) curves, which includes both smooth and polygonal curves, as a way of unifying the treatment of curvature. Here we briefly review the theory of smooth curves from the point of view we will later adopt for surfaces.

The curvatures of a smooth curve γ (which we usually assume is parametrized by its arclength s) are the local properties of its shape invariant under Euclidean motions. The only first-order information is the tangent line; since all lines in space are equivalent, there are no first-order invariants. Second-order information is given by the osculating circle, and the invariant is its curvature $\kappa = 1/r$.

For a plane curve given as a graph $y = f(x)$ let us contrast the notions of curvature and second derivative. At a point p on the curve, we can find either one by translating p to the origin, transforming so the curve is horizontal there, and then comparing to a standard set of reference curves. The difference is that for curvature, the transformation is a Euclidean rotation, while for second derivative, it is a shear $(x, y) \mapsto (x, y - ax)$. A parabola has constant second derivative f'' because it looks the same at any two points after a shear. A circle, on the other hand, has constant curvature because it looks the same at any two points after a rotation.

A plane curve is completely determined (up to rigid motion) by its (signed) curvature $\kappa(s)$ as a function of arclength s. For a space curve, however, we need to look at the third-order invariants; these are the torsion τ and the derivative κ', but the latter of course gives no new information. These are now a complete set of invariants: a space curve is determined by $\kappa(s)$ and $\tau(s)$.

Generically speaking, while second-order curvatures suffice to determine a hypersurface (of codimension 1), higher-order invariants are needed for higher codimension. For curves in $\mathbb{E}^d$, for instance, we need $d-1$ generalized curvatures, of order up to n, to characterize the shape.

Let us examine the case of space curves $\gamma \subset \mathbb{E}^3$ in more detail. At every point $p \in \gamma$ we have a splitting of the tangent space $T_p\mathbb{E}^3$ into the tangent line $T_p\gamma$ and the normal plane. A framing along γ is a smooth choice of a unit normal vector N_1, which is then completed to the oriented orthonormal frame (T, N_1, N_2) for $\mathbb{E}^3$, where $N_2 = T \times N_1$.

Taking the derivative with respect to arclength, we get a skew-symmetric matrix (an infinitesimal rotation) describing how the frame changes:

$$\begin{pmatrix} T \\ N_1 \\ N_2 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -kappa_2 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N_1 \\ N_2 \end{pmatrix}.$$

Here, $T'(s) = \sum \kappa_i N_i$ is the *curvature vector* of γ, while τ measures the twisting of this framing.

If we choose N_1 at a basepoint along γ a natural choice of framing is the *parallel framing* or *Bishop framing* [Bis75] defined by the condition $\tau = 0$. Equivalently, the vectors N_i are parallel-transported along γ from the basepoint, using the Riemannian connection induced by the immersion in $\mathbb{E}^3$. One should note that this is not necessarily a closed framing along a closed loop γ; when we return to the basepoint, N_1 has been rotated through an angle called the writhe of γ.

Other framings are also often useful. For instance, if γ lies on a surface M with unit normal ν, it is natural to choose $N_1 = \nu$. Then $N_2 = \eta := T \times \nu$ is called the *cornormal* vector, and (T, ν, η) is the *Darboux frame* (adapted to $\gamma \subset M \subset \mathbb{E}^3$). The curvature vector of γ decomposes into parts tangent and normal to M as $T' = \kappa_g \eta + \kappa_n \nu$. Here, κ_n measures the *normal curvature* of M in the direction T, and is independent of γ, while κ_g, the *geodesic curvature* of γ in M, is an intrinsic notion, unchanged if we isometrically deform the immersion of M into space.

When the curvature vector of γ never vanishes, we can write it as $T' = \kappa N$ where $\kappa > 0$ and N is a unit vector, the *principal normal*. This yields the orthonormal *Frenet frame* (T, N, B), whose twisting τ is the torsion of γ.

The *total curvature* of a smooth curve is $\int \kappa\, ds$. In [Sul06] we investigated a number of results: For closed curves, the total curvature is at least 2π (Fenchel) and for knotted space curves the total curvature is at least 4π (Fáry/Milnor). For plane curves, we can consider instead the signed curvature, and find that $\int \kappa\, ds$ is always an integral multiple of 2π. Then (following Milnor) we defined the total curvature of a polygonal curve simply to be the sum of the turning angles at the vertices. Then all these theorems on total curvature remain true. Our goal, when defining curvatures for polyhedral surfaces, will be to again ensure that integral relations remain exactly true.

2. Curvatures of Smooth Surfaces

Given a (two-dimensional, oriented) surface M (smoothly immersed) in $\mathbb{E}^3$, we understand its local shape by looking at the Gauß map $\nu : M \to \mathbb{S}^2$ given by the unit normal vector $\nu = \nu_p$ at each point $p \in M$. Its derivative at p is a linear map from T_pM to $T_{\nu_p}\mathbb{S}^2$. But these spaces are naturally identified, being parallel planes in $\mathbb{E}^3$, so we can view the derivative as an endomorphism $-S_p : T_pM \to T_pM$. The map S_p is called the shape operator (or Weingarten map).

The shape operator is the complete second-order invariant (or curvature) which determines the original surface M. (This statement has been left intentionally a bit vague, since without a standard parametrization like arclength, it is not quite clear how one should specify a function along an unknown surface.) Usually, however, it is more convenient not to work with the operator S_p but instead with scalar quantities. Its eigenvalues κ_1 and κ_2 of S_p are called principal curvatures, and (since they cannot be globally distinguished) it is their symmetric functions which have the most geometric meaning.

We define the *Gauß curvature* $K := \kappa_1\kappa_2$ as the determinant of S_p and the *mean curvature* $H := \kappa_1 + \kappa_2$ as its trace. Note that the sign of H depends on the choice of unit normal ν, and so often it is more natural to work with the *vector mean curvature* (or mean curvature vector) $\mathbf{H} := H\nu$. Note furthermore that some authors use the opposite sign on S_p and thus H, and many use $H = (\kappa_1 + \kappa_2)/2$, justifying the name *mean* curvature. Our conventions mean that the mean curvature vector for a convex surface points inwards (like the curvature vector for a circle). For a unit sphere oriented with inward normal, the Gauß map ν is the antipodal map, $S_p = I$, and $H = 2$.

The Gauß curvature is an intrinsic notion, depending only on the pullback metric on the surface M, and not on the immersion into space. That is, K is unchanged by bending the surface without stretching it. For instance, a developable surface like a cylinder or cone has $K = 0$ because it is obtained by bending a flat plane. One intrinsic definition of $K(p)$ is obtained by comparing the circumferences C_ε of (intrinsic) ε-balls around p to the value $2\pi\varepsilon$ in $\mathbb{E}^2$. We get

$$\frac{C_\varepsilon}{2\pi\varepsilon} = 1 - \frac{\varepsilon^2}{6}K + \mathcal{O}(\varepsilon^3).$$

Mean curvature is certainly not intrinsic, but it has a nice variational interpretation. Consider a variation vectorfield V on M, compactly supported away from any boundary. Then $H = -\delta\,\mathrm{Area}/\delta\,\mathrm{Vol}$ in the sense that

$$\delta_V\,\mathrm{Vol} = \int V \cdot \nu\, dA, \quad \delta_V\,\mathrm{Area} = -\int V \cdot H\nu\, dA.$$

With respect to the L^2 inner product $\langle U, V\rangle := \int U_p \cdot V_p \, dA$ on vectorfields, the vector mean curvature is the negative gradient of the area functional, often called the first variation of area: $\mathbf{H} = -\nabla\,\mathrm{Area}$. (Similarly, the negative gradient of length for a curve is its curvature vector κN.)

Just as κ is the geometric version of second derivative for curves, mean curvature is the geometric version of the Laplacian Δ. Indeed, if a surface M is written locally as the graph of a height function f over its tangent plane T_pM then $H(p) = \Delta f$. Alternatively, we can write $\mathbf{H} = \nabla_M \cdot \nu = \Delta_M \mathbf{x}$, where $\mathbf{x}$ is the position vector in $\mathbb{R}^3$ and Δ_M is Beltrami's surface Laplacian.

If we flow a curve or surface to reduce its length or area, by following these gradients κN and $H\nu$, the resulting parabolic heat flow is slightly nonlinear in a natural geometric way. This so-called *mean-curvature flow* has been extensively studied as a geometric smoothing flow.

3. Integral Curvature Relations for Surfaces

For surfaces, the integral curvature relations we want to consider relate area integrals over a region $D \subset M$ to arclength integrals over the boundary $\gamma = \partial D$. The Gauß/Bonnet theorem says, when D is a disk,

$$2\pi - \iint_D K\, dA = \oint_\gamma \kappa_g \, ds = \oint_\gamma T' \cdot \eta \, ds = -\oint \eta' \cdot \, d\mathbf{x},$$

where $d\mathbf{x} = T\,ds$ is the vector line element along γ. This implies that the total Gauß curvature of D depends only on a collar neighborhood of γ: if we make any modification to D supported away from the boundary, the total curvature is unchanged (as long as D remains topologically a disk). We will extend the notion of Gauß curvature from smooth surfaces to more general surfaces (in particular polyhedral surfaces) by requiring this property to remain true.

The other relations are all proved by Stokes' Theorem, and thus only depend on γ being the boundary of D in a homological sense; for these D is not required to be a disk. First consider the vector area

$$\mathbf{A}_\gamma := \tfrac{1}{2} \oint_\gamma \mathbf{x} \times d\mathbf{x} = \iint_D \nu \, dA.$$

The right-hand side represents the total vector area of any surface spanning γ, and the relation shows this to depend only on γ (and this time not even on a collar neighborhood). The integrand on the left-hand side depends on a choice of origin for the coordinates, but because we integrate over a closed loop, the integral is independent of this choice. Both sides of this vector area formula can be interpreted directly for a polyhedral surface, and the equation remains true in that case. We note also that this vector area $\mathbf{A}_\gamma$ is preserved when γ evolves under the Hasimoto or smoke-ring flow.

A simple integral for a curve γ from p to q says that

$$T(q) - T(p) = \int_p^q T'(s)\,ds = \int \kappa N\,ds.$$

This can be viewed as a balance between tension forces trying to shrink the curve, and sideways forces holding it in place. It is the relation used in proving that the vector curvature κN is the first variation of length.

The analog for a surface patch D is the mean curvature force-balance equation

$$\oint_\gamma \eta\,ds = -\oint_\gamma \nu \times d\mathbf{x} = \iint_D H\nu\,dA = \iint_D \mathbf{H}\,dA.$$

Again this represents a balance between surface tension forces acting in the conormal direction along the boundary of D and what can be considered as pressure forces (espcially in the case of constant H) acting normally across D. We will use this equation to develop the analog of mean curvature for discrete surfaces.

Two other similar relations that we will not need later are the torque balance

$$\oint_\gamma \mathbf{x} \times \eta\,ds = \oint_\gamma \mathbf{x} \times (\nu \times d\mathbf{x}) = \iint_D H(\mathbf{x} \times \nu)\,dA = \iint_D \mathbf{x} \times \mathbf{H}\,dA$$

and the area relation

$$\oint_\gamma \mathbf{x} \cdot \eta\,ds = \oint_\gamma \mathbf{x} \cdot (\nu \times d\mathbf{x}) = \iint_D (\mathbf{H} \cdot \mathbf{x} - 2)\,dA.$$

4. Discrete Surfaces

For us, a discrete or polyhedral surface $M \subset \mathbb{R}^3$ will mean a triangulated surface with a PL map into space. In more detail, we start with an abstract combinatorial triangulation—a simplicial complex—representing a 2-manifold with boundary. We then pick positions $p \in \mathbb{R}^3$ for every vertex, which uniquely determine a linear map on each triangle; these fit together to form the PL map.

4.1. Gauß curvature

It is well known how the notion of Gauß curvature extends to such discrete surfaces M. Any two adjacent triangles (or, more generally, any simply connected region in M not including any vertices) can be flattened—developed isometrically into the plane. Thus the Gauß curvature is supported on the vertices $p \in M$. In fact, to keep the Gauß/Bonnet theorem true, we must take

$$\iint_D K\,dA := \sum_{p \in D} K_p; \qquad K_p := 2\pi - \sum_i \theta_i.$$

Here, the angles θ_i are the interior angles at p of the triangles meeting there, and K_p is often known as the angle defect at p. If D is any neighborhood of p contained in $\mathrm{Star}(p)$, then $\oint_{\partial D} \eta\,ds = \sum \theta_i$; when the triangles are acute, this is most easily seen by letting ∂D be the path connecting their circumcenters and crossing each edge perpendicularly.

(Similar arguments lead to a notion of Gauß curvature—defined as a measure—for any rectifiable surface. For our polyhedral surface, this measure consists of point masses at vertices. Surfaces can also be built from intrinsically flat pieces joined along curved edges. Their Gauß curvature is spread out with a linear density along these edges. This technique is often used in designing clothes, where corners would be undesirable.)

Note that K_p is clearly an intrinsic notion, as it should be, depending only on the angles of each triangle and not on the precise embedding into $\mathbb{R}^3$. Sometimes it is useful to have a notion of combinatorial curvature, independent of all geometric information. Given just a combinatorial triangulation, we can pretend that each triangle is equilateral with angles $\theta = 60°$, whether or not that geometry could be embedded in space. The resulting *combinatorial curvature* is $K_p = \frac{\pi}{3}(6 - \deg p)$. In this context, the global form $\sum K_p = 2\pi\chi(M)$ of Gauß/Bonnet amounts to nothing more than the definition of the Euler characteristic χ.

4.2. Vector area

The vector area formula

$$\mathbf{A}_\gamma := \tfrac{1}{2}\oint_\gamma \mathbf{x} \times d\mathbf{x} = \iint_D \nu\, dA$$

needs no special interpretation for discrete surfaces: both sides of the equation make sense directly, since the surface normal ν is well-defined almost everywhere. However, it is worth interpreting this formula for the case when D is the star of a vertex p. More generally, suppose γ is any closed curve (smooth or polygonal), and D is the cone from p to γ (the union of all line segments pq for $q \in \gamma$). Fixing γ and letting p vary, we find that the volume enclosed by this cone is a linear function of p, and $\mathbf{A}_p := \nabla_p \operatorname{Vol} D = \mathbf{A}/3 = \frac{1}{6}\oint_\gamma \mathbf{x} \times d\mathbf{x}$. We also note that any such cone D is intrinsically flat except at the cone point p, and that $2\pi - K_p$ is the cone angle at p.

4.3. Mean curvature

The mean curvature of a discrete surface M is supported along the edges. If e is an edge, and $e \subset D \subset \operatorname{Star}(e) = T_1 \cup T_2$, then

$$\mathbf{H}_e := \iint_D \mathbf{H}\, dA = \oint_{\partial D} \eta\, ds = e \times \nu_1 - e \times \nu_2 = J_1 e - J_2 e.$$

Here ν_i is the normal vector to the triangle T_i, and J_i is rotation by $90°$ in the plane of that triangle. Note that $|\mathbf{H}_e| = 2|e| \sin \frac{\theta_e}{2}$ where θ_e is the exterior dihedral angle along the edge, defined by $\cos\theta_e = \nu_1 \cdot \nu_2$.

No nonplanar discrete surface has $\mathbf{H}_e = 0$ along every edge. But this discrete mean curvature can cancel out around the vertices. We set

$$2\mathbf{H}_p := \sum_{e \ni p} \mathbf{H}_e = \iint_{\operatorname{Star}(p)} \mathbf{H}\, dA = \oint_{\operatorname{Link}(p)} \eta\, ds.$$

The area of the discrete surface is a function of the vertex positions; if we vary only one vertex p, we find that $\nabla_p \operatorname{Area}(M) = -\mathbf{H}_p$.

Suppose that vertices adjacent to p are $p_1, \ldots, p_n$. Then we have

$$3\mathbf{A}_p = 3\nabla_p \operatorname{Vol} = \iint_{\operatorname{Star} p} \nu \, dA$$
$$= \tfrac{1}{2} \oint_{\operatorname{Link} p} \mathbf{x} \times d\mathbf{x} = \tfrac{1}{2} \sum_i p_i \times p_{i+1}$$

and similarly

$$2\mathbf{H}_p = \sum \mathbf{H}_{pp_i} = -2\nabla_p \operatorname{Area} = \sum J_i(p_{i+1} - p_i)$$
$$= \sum_i (\cot \alpha_i + \cot \beta_i)(p - p_i),$$

where α_i and β_i are the angles opposite edge pp_i in the two incident triangles.

Note that if we change the combinatorics of a discrete surface M by introducing a new vertex p along an existing edge e, and subdividing the two incident triangles, then $\mathbf{H}_p$ in the new surface equals the original $\mathbf{H}_e$, independent of where along e we place p. This allows a variational interpretation of $\mathbf{H}_e$.

4.4. Minkowski mixed volumes

A somewhat different interpretation of mean curvature for convex polyhedra is suggested by Minkowski's theory of mixed volumes (which actually dates in this form well earlier). If X is a smooth convex body in $\mathbb{R}^3$ and $B_t(X)$ denotes its t-neighborhood, then

$$\operatorname{Vol}(B_t(X)) = \operatorname{Vol} X + t \operatorname{Area} X + \frac{t^2}{2} \int_X H \, dA + \frac{t^3}{3} \int_X K \, dA.$$

Here, the last integral is always 4π.

When X is instead a convex polyhedron, the only term that needs a new interpretation is $\int_X H \, dA$. The correct replacement for this term is then $\sum_e |e| \, \theta_e$. This suggests $H_e := |e| \, \theta_e$ as a notion of total mean curvature for the edge e.

We note the difference between this formula and our earlier $|\mathbf{H}_e| = 2|e| \sin \theta_e/2$. Either one can be derived by replacing the edge e with a sector of a cylinder of length $|e|$ and arbitrary (small) radius r. We find then

$$\iint \mathbf{H} \, dA = \mathbf{H}_e, \qquad \iint H \, dA = H_e.$$

The difference is explained by the fact that one formula integrates the scalar mean curvature while the other integrates the vector mean curvature.

4.5. CMC surfaces and Willmore surfaces

A smooth surface which minimizes area under a volume constraint has constant mean curvature; the constant H can be understood as the Lagrange multiplier for the constrained minimization problem. A discrete surface which minimizes area among surfaces of fixed combinatorial type and fixed volume will have constant discrete mean curvature H in the sense that at every vertex, $\mathbf{H}_p = H\mathbf{A}_p$, or equivalently $\nabla_p \operatorname{Area} = -H\nabla_p \operatorname{Vol}$. In general, of course, the vectors $\mathbf{H}_p$ and $\mathbf{A}_p$ are not even parallel: they give two competing notions of a normal vector at p.

Still,

$$h_p := \frac{|\nabla_p \operatorname{Area}|}{|\nabla_p \operatorname{Vol}|} = \frac{|\mathbf{H}_p|}{|\mathbf{A}_p|} = \frac{|\iint_{\operatorname{Star} p} \mathbf{H}\, dA|}{|\iint_{\operatorname{Star} p} \nu\, dA|}$$

gives a better notion of mean curvature near p than, say, the smaller quantity $3|\mathbf{H}_p|/\operatorname{Area}(\operatorname{Star}(p)) = |\iint \mathbf{H}\, dA|/\iint 1\, dA$.

For this reason, a good discretization of the Willmore elastic energy $\iint H^2\, dA$ is given by $\sum_p h_p^2 \frac{1}{3} \operatorname{Area}(\operatorname{Star}(p))$.

4.6. Relation to discrete harmonic maps

Discrete minimal surfaces minimize area, but also have other properties similar to those of smooth minimal surfaces. For instance, in a conformal parameterization, their coordinate functions are harmonic. We don't know when in general a discrete map should be considered conformal, but the identity map is certainly conformal. We have that M is discrete minimal if and only if $\operatorname{Id} : M \to \mathbb{R}^3$ is discrete harmonic. Here a PL map $f : M \to N$ is called discrete harmonic if it is a critical point for the Dirichlet energy $E(f) := \sum_T |\nabla f_T|^2 \operatorname{Area}_M(T)$. We find that $E(f) - \operatorname{Area} N$ is a measure of nonconformality. For the identity map, $E(\operatorname{Id}_M) = \operatorname{Area}(M)$ and $\nabla_p E(\operatorname{Id}_M) = \nabla_p \operatorname{Area}(M)$ confirming that M is minimal if and only if Id_M is harmonic.

5. Vector Bundles on Polyhedral Manifolds

Here we give a general definition of for vector bundles and connections on polyhedral manifolds; this leads to another interpretation of the Gauß curvature for a polyhedral surface.

Here, a polyhedral n-manifold P^n will mean a CW-complex which is homeomorphic to an n-dimensional manifold, and which is regular and satisfies the intersection condition (see [Zie06]). That is, each facet (n-cell) is embedded in P_n with no identifications on its boundary, and the intersection of any two faces (of any dimension) is a single face (if nonempty).

Definition. A *discrete* rank k *vector bundle* V^k over P^n consists of a vectorspace $V_f \cong \mathbb{R}^k$ for each facet f of P. A *connection* on V^k is a choice of isomorphism ϕ_r between V_f and $V_{f'}$ for each ridge (or $(n-1)$–cell) of P. Here, the ridge r is the intersection of the two facets f and f'. We are most interested in the case where the vectorspaces V_f have inner products, and the isomorphisms ϕ_r are orthogonal.

Consider the case $n = 1$, where P is a polygonal curve. On an arc (or open curve) any vector bundle is trivial. On a loop (or closed curve), a rank k vector bundle is determined (up to isomorphism) simply by its holonomy around the loop, an automorphism $\phi : \mathbb{R}^k \to \mathbb{R}^k$.

Now suppose P^n is linearly immersed in $\mathbb{E}^d$ for some d. That is, each k-face of P is mapped homeomorphically to a convex polytope in an affine k-plane in $\mathbb{E}^d$, and the combinatorial star of each vertex is embedded.

Then it is clear how to define the discrete tangent bundle T (of rank n) and normal bundle N (of rank $d-n$). Namely, T_f is the n-plane parallel to the affine hull of f, and N_f is the orthogonal $(d-n)$–plane. These do inherit inner products from the euclidean structure of $\mathbb{E}^d$.

There are also natural analogs of the Levi-Civita connections on these bundles. Namely, for each ridge r, let $\alpha_r \in [0,\pi)$ be the exterior dihedral angle bewteen the facets f_i meeting along r. (Our assumption that P is immersed is what rules out $\alpha_r = \pi$, which would cause trouble below.) And let $\phi_r : \mathbb{E}^d \to \mathbb{E}^d$ be the simple rotation by this angle, fixing the affine hull of r (and the space orthogonal to the affine hull of the f_i). We see that ϕ_r restricts to give maps $T_f \to T_{f'}$ and $N_f \to N_{f'}$; these form the connections we want. (Note that $T \oplus N = \mathbb{R}^d$ is a trivial vector bundle over P^n, but the maps ϕ_r give a nontrivial connection on it.)

Consider again the example of a closed polygonal curve $P^1 \subset \mathbb{E}^d$. The rank-1 tangent bundle has trivial holonomy. The holonomy of the normal bundle is some rotation of $\mathbb{E}^{d-1}$. For $d=3$ this rotation of the plane is specified by an angle equal (modulo 2π) to the writhe of P^1. (To define the writhe of a curve as a real number, rather than just modulo 2π, requires a bit more care, and requires the curve to be embedded.)

Next consider a two-dimensional polyhedral surface P^2 and its tangent bundle. Around a vertex v we can compose the ring of isomorphisms ϕ_e across the edges incident to v. This gives a self-map $\phi_v : T_f \to T_f$ which we see is a rotation of the tangent plane by an angle equal to the Gauß curvature K_v.

Now consider the general case of the tangent bundle to a polyhedral manifold $P^n \subset \mathbb{E}^d$. Suppose v is a codimension-2 face of P. Then composing the ring of isomorphisms across the ridges incident to v gives an automorphism of $\mathbb{E}^n$ which is the local holonomy, or *curvature* of the Levi-Civita connection around v. We see that this is a rotation fixing the affine hull of v. To define the curvature (which can be interpreted as a sectional curvature in the 2-plane normal to v) as a real number and not just modulo 2π, we should look again at the angle defect around v, which is $2\pi - \sum \beta_i$ if the β_i are the interior dihedral angles along v of the facets f incident to v.

In the case of a hypersurface P^{d-1} in $\mathbb{E}^d$, the one-dimensional normal bundle is locally trivial: there is no curvature or local holonomy around any v. Globally, the normal bundle is of course trivial exactly when P is orientable.

References

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