Methodology for a system to support network time coordination of services at transfer nodes

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LIST OF SYMBOLS USED

a_i	[min]	value of the maximum permissible time shift of connections on a coordinated line $i \in L$,
$\bar{a}_i$	[min]	average value of the maximum permissible time shift on the coordinated line $i \in L$ when an alternating headway is applied on it,
e_i	[min]	elementary time unit in terms of time coordination on a coordinated line $i \in L$,
h_{uilkjs}	[min]	a non-negative variable modelling the time loss of each passenger transferring at the transfer node $u \in U$ from a connection $k \in P_{il}$ of line $i \in L_u$ travelling in the direction of $l \in S$ to the nearest line connection $j \in L_u$ going in the direction of $s \in S$,
I		the set of passengers using public transport,
K_p	[min]	length of the coordination period,
L_u		set of lines coordinated at the transfer node $u \in U$,
L_u^*		the set of outgoing lines from the transfer node $u \in U$ on which the alternating headway is applied,
L_u^{**}		the set of incoming lines to the transfer node $u \in U$on which the alternating headway is applied,
L_{uj}		the set of incoming lines to the transfer node $u \in U$ from which passengers transfer to the outgoing line $j \in L_u$,
n_{ui}		number of line connections $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period,
M		an enough high constant
N		number of connections in the alternating headway applied between the connections of the coordinated line $i \in L_u$ in a partial coordination period,
R_0^+		set of non-negative real numbers,

S_i		set of coordinated line directions $i \in L$ (if all coordinated lines have the same number of directions, it is possible to use the simplified designation S),
$t_č$	[min]	the passenger's waiting time for the nearest connection at the first stop on the route to the final destination,
$t_{č_i}$	[min]	passenger waiting time $i \in I$ for the nearest connection at the first stop on the route to the final destination,
t_{dp}	[min]	the total time of the passenger's stay in the means of transport (journey time) on public transport routes from the start of the journey until reaching the final destination,
t_{dp_i}	[min]	the total duration of the passenger's stay $i \in I$ in the means of transport (passenger's transport time $i \in I$) on public transport routes from the start of the journey until reaching the final destination,
t_p	[min]	the total transport time of the passenger from the start of the journey until reaching the final destination,
t_{p_i}	[min]	the total transfer time of the passenger $i \in I$ from the start of the journey until reaching the final destination,
t_{pre}	[min]	the total time of passenger transfers on the route of the used public transport lines (passenger walking time and passenger waiting time for the transfer),
t_{pre_i}	[min]	the total time of the passenger's transfers $i \in I$ on the route of the used public transport lines (passenger walking time $i \in I$ and passenger waiting time $i \in I$ for the service at the transfer node),
$tprest_{uij}$	[min]	the value of the passenger transfer time between the stations of the incoming line $i \in L_u$ and the point of departure of the departing line $j \in L_u$ at the transfer node $u \in U$,
t_{ui1}		the earliest possible service time of the transfer node $u \in U$ by the first connection of the incoming line $i \in L_u$ in the coordination period,

t_{ujn_j}		the earliest possible service time of the transfer node $u \in U$ by the last connection of the departing line $j \in L_u$ in the coordination period,
t_{uj1}		the earliest possible service time of the transfer node $u \in U$ by the first connection of the departing line $j \in L_u$ in the coordination period,
t_1	[min]	the walking time of the passenger from the start of the journey to the first boarding stop on his/her route,
t_{1_i}	[min]	passenger walking time $i \in I$ from the start of the journey to the first public transport stop on the route,
t_2	[min]	the passenger's walking time from the last public transport stop on his/her route to the final destination,
t_{2_i}	[min]	passenger walking time $i \in I$ from the last stop of public transport on his/her route to the final destination,
T_i	[min]	the value of the constant headway applied between the connections of the line $i \in L_u$,
$\overline{T_i}$	[min]	average value of the headway in the partial coordination period of the line $i \in L$ on which the alternating headway is applied,
T_i^p	[min]	value of the partial coordination period of the line $i \in L_u$on which the alternating headway is applied,
U		set of transfer nodes,
v_{il}		auxiliary binary variable introduced to increase the value of the base headway by α elementary time units after each even line connection $i \in L_u^{**} \cup L_u^*$ in the direction of $l \in S$ within the coordination period serving the last transfer node on the line's route,
w_{il}		auxiliary binary variable introduced to increase the value of the base headway by α elementary time units after each odd line connections $i \in L_u^{**} \cup L_u^*$ in the direction of $l \in S$ within the coordination period, except for the first line connection serving the last transfer node on the line's route,

x_{il}	[min]	non-negative variable modelling the time shift of all connections of the line $i \in L$ in the direction $l \in S$ calculated from their earliest possible time positions,
$z_{uilkjsp}$		auxiliary binary variable modelling the formation of the coordination bond between the connection $k \in P_{il}$ of the incoming line $i \in L_u$ travelling in the direction of $l \in S$ and the connection $p \in P_{js}$ of the outgoing line $j \in L_u$ going in the direction of $s \in S$ at the transfer node $u \in U$,
Z_0^+		the set of non-negative integers.

CONTENTS

1 OVERVIEW OF TERMS USED IN THE TEXT OF THE METHODOLOGY

Line	the set of connections that provide regular transport to certain places [1].
Connection	a timetabled or otherwise scheduled and locally determined transport service to specific locations within regular transport operation [1].
Stop	a place on a transport route marked and equipped in a prescribed manner, intended in particular for the boarding and alighting of passengers [1].
Final stop	a public transport stop located at the beginning and end of the route of each line [1].
Departure stop	the stop served on the route of the connection as the first in the sequence (the stop at which the vehicle operating the connection starts its journey).
Final destination stop	the final stop served on the route of the connection as the last in the sequence (the stop where the vehicle operating the connection ends its journey).
Transfer node	a significant stop on a public transport network served by at least two lines of the same or different modes, where there is a significant flow of transferring passengers between lines of the same or different modes, the transfer node may be the final stop of the line.

Incoming line	a line from which passengers alight at the transfer node in order to use another line immediately afterwards.
Outbound line	a line whose passengers board at the transfer node immediately after having previously used a connection of another line.
Transfer time	the minimum time interval required to ensure the exit of passengers from the vehicle of the inbound connection, the transfer from the connection's arrival point to the connection's departure point and the boarding of passengers onto the vehicle of the departing connection (the time interval may also take into account the time reserve in the event of a delay of the arriving connection).
Network time coordination	a computational process whereby connections on different lines whose routes converge at the same transfer node(s) (nodal coordination) or whose routes are at least partially identical on one common section (section coordination) are shifted in time to make the public transport offer more attractive according to the selected optimisation criterion.
Headway (line interval)	a periodically recurring time interval in which connections of a given line are operated in a given direction according to the timetable; the headway may be constant or alternating.

Constant headway	A headway on the line whose value does not change between every two consecutive connections of the line during the coordination period.
Alternating headway	two or more different headway values applied between two consecutive connections on a line, the values being repeated periodically in the same sequence.
Basic headway	the minimum of the headway values used in alternating interval operating conditions.
Coordination period	a continuous day/night period, or part thereof, during which the same operating conditions are established on the coordinated lines in terms of the headway rates used.
Coordination period	the time interval (usually a sub-period of the coordination period) during which network-wide time coordination is performed.
Coordination link	a specific requirement to provide for the transfer of passengers between vehicles of the same mode or different modes of transport defined by the number of the transfer node, the number of the line, direction and connection of the line arriving at the transfer node, the number of the line and direction of the line departing from the transfer node and the average number of passengers transferring between the connections of the two lines per coordination period or the whole coordination period.

Nodal coordination period	the time periods (usually a sub-interval of the coordination period) in which the nodal network time coordination is performed within a node (its value can be the same as the value of the coordination period in the whole network).
Partial coordination period for the line	a time period occurring only in the case of an alternating headway and containing all periodically repeating values of the alternating headway (a coordination period may consist of one or more sub-coordination periods).
Internal calendar	time counting rules in which the coordination period does not start in real time, but always starts at a time of 0.
Travel time	the time that elapses between the time of the regular departure of the connection from the designated stop and the time of the regular arrival at another (e.g. neighbouring) stop of the connection.

2 INTRODUCTION

2.1 Subject and aim of the methodology

The subject of the methodology is to provide a comprehensive view of the problem of solving network time coordination of connections in transfer nodes (hereinafter referred to as "network node time coordination"). Network node time coordination also includes making links between the connections of coordinated lines.

The aim of the presented methodology will be to familiarize the professional public with the application potential and examples of practical applications of mixed integer linear programming methods in solving network nodal time coordination problems.

2.2 Characteristics of the current situation and justification of the innovation of the procedures contained in the methodology

The current situation in the Czech Republic and Slovakia is characterised mainly by the manual creation of timetables without a more fundamental and systematic use of optimisation calculations. For the design of the time positions of public transport connections, there are certain computational aids available, used by designers and constructors of timetables in the environment of individual public transport organisers or carriers, created e.g. in MS EXCEL and used e.g. in network section time coordination, but their optimality is not guaranteed for the results achieved by them. No computational aid is currently available for network node time coordination.

Although some approaches to network node time coordination are known from the foreign literature, they are usually limited to coordination problems related to the last connections listed in the timetables, and no systematic computational approach has been found that could design timetables guaranteeing optimality. No information was found on the optimisation of time loss of passengers changing at transfer nodes under alternating headway conditions.

The innovation of the methodology lies in the fact that it is the first methodology dedicated to the problem of network node time coordination using optimization methods to solve the problem. The text of the methodology contains guidelines for solving problems related to network node time coordination under constraints of a wide range of operational variants differing in the types of headways that may occur in real operation. The use of optimization methods will allow the solvers to reach the optimum (global, local).

An optimization method that is suitable for solving network node time coordination problems is mathematical programming, specifically one part of it, namely mixed integer linear programming. The advantage of mixed integer linear programming is that it is a general optimization approach converging to a global optimal solution, and also that there is a wide range of general-purpose software tools (solvers) that allow solving of even large-scale problems, i.e., they allow addressing network node time coordination problems with a significant number of required coordination links.

2.3 Users of the methodology

There are five basic categories of users for whom the results of the methodology may be relevant.

The primary category of users of the project results contained in the methodology are timetable designers, who can directly use the methodology in their work to reduce the time loss of passengers transferring between different public transport lines at transfer nodes.

The second category of users of the project results contained in the methodology are managers of transport companies working at all levels of management and employees of public transport organisers or other contracting authorities (e.g. authorised employees of municipal authorities) who can use it in conceptual planning of public transport development.

The third category of users of the project results contained in the methodology are customers of public transport (passengers), for whom an attractive offer of public transport will be created.

The fourth category of users of the project results contained in the methodology are researchers working in public and private research institutions, who can use it in further development of the issue of effective organization of public transport.

The fifth category of users of the project results contained in the methodology are teachers and students of universities, who can use it in the pedagogical process focused on the issues of effective organization of public transport.

3 SUMMARY OF BASIC KNOWLEDGE ABOUT MIXED INTEGER LINEAR PROGRAMMING NEEDED TO SOLVE OPTIMIZATION PROBLEMS CONCERNING NETWORK NODE TIME COORDINATION

The solution of network node time coordination problems using mixed integer linear programming methods, analogous to the solution of any other linear programming optimization problem, consists of two basic phases, namely the phase devoted to the creation of the mathematical model and the phase devoted to its solution. In the phase of mathematical model creation, it is mainly about the formulation of the optimization problem (identification of quantities that do not change during the calculation, identification of expected decisions and identification of the optimization criterion) and writing the model by mathematical means.

The problem formulation results in two basic groups of quantities, namely quantities whose values do not change during the optimization calculation (so-called constants) and quantities whose values change during the optimization calculation (so-called variables). Variables usually model the expected decisions and we make the necessary decisions based on their values after the optimization calculation is completed.

Each quantity (whether constant or variable) used in the mathematical model of mixed integer linear programming must be labelled in some way in the mathematical model. The designation of the quantities depends on the solver; in connection with the choice of the designation of the quantities, it can only be stated that the designation of the quantities should be as simple as possible and should also have, if possible, an appropriate predictive power.

Every mathematical model of linear programming is composed of two basic parts, namely an optimization criterion and a set of constraints.

An optimization criterion is a quantity whose value characterizes the quality of a specific (admissible) solution from the perspective of the declared interest of the optimization problem's submitter. If the optimization criterion can be expressed by a function, the term optimization criterion is replaced by the terms objective function or criterial function. As the optimization criterion will be expressed by a functional prescription in the present methodology, the term objective function will be used in the text of the methodology. The objective function must be chosen carefully to represent the legitimate interest of the submitter.

An integral part of the objective function is also the requirement for the type of its extreme. The value of the objective function can be either maximized or minimized. The type of extreme chosen depends on the nature of the objective function. There are also optimization problems with multiple objective functions, which are called multi-criteria (multicriteria) optimization problems. However, since the mathematical models in the text of the methodology will not contain more than one objective function, the details of the multi-criteria optimization methods will not be addressed in the text of the methodology.

The set of constraints refers to the limiting factors of the optimization problem that must be accepted during the optimization calculation. Sometimes the term restrictive conditions is also used instead of constraints. Acceptance of limiting factors is directly related to the admissibility of the solution that is the result of the optimization calculation, therefore we also say that the set of constraints defines the set of admissible solutions of the problem during the optimization calculation.

Some constraints may express limiting factors that seem obvious at first glance, however, the mathematical model must include the obvious factors, because even what seems obvious to the solver from a real-world perspective must be incorporated mathematically into the mathematical model.

Constraints are of two types - obligatory and structural.

Obligatory constraints represent the domains of definition (sometimes the term value domains is also used in this context) of the variables used in the mathematical model. In mixed integer linear programming, three types of domains of definition are used, namely the set of non-negative real numbers, denoted in the following text by the symbol R_0^+, the set of non-negative integers, denoted in the following text by the symbol Z_0^+ and the set of values 0 and 1, denoted in the following text by $\{0; 1\}$. The choice of the domain of definition of the variables is made depending on the type of decision the variable models. In some cases, multiple types of domains of definition can be used for a particular group of variables. If a domain of definition R_0^+, is included among the possible types of domains of definition then it is always preferred over the others because it significantly affects the flow (time) of the optimization calculation.

Structural constraints perform two functions during the optimization calculation. Either they represent real constraints that influence the process of finding a solution (in the conditions of the solved tasks, e.g. the values of the maximum allowed time shifts of coordinated lines) or they create logical links between variables modelling the respective decisions. Structural constraints that create logical links between variables are therefore sometimes also called binding constraints. Structural constraints in mathematical models take the form of equations or inequalities with the occurrence of relational signs ($\leq; \geq; =$).

In terms of creating a mathematical model of mixed integer linear programming, it is also necessary to state three basic rules for counting with variables. Expressions containing variables can only be added, subtracted or multiplied by a real constant in the mathematical model of mixed integer linear programming.

There is no clear guidance for the creation of a mathematical model of mixed integer linear programming, there are only certain rules of a more general nature that are recommended to follow [2]:

1. the optimization criterion is analysed in terms of the expected decisions on which its value depends, appropriate variables are selected in the context of the analysis, including their domains of definition, and an objective function is constructed;

2. the individual limiting factors are analysed in turn and expressed using equations or inequalities with relational signs containing constants and introduced variables, additional variables are introduced if required and additional relationships between variables are added as necessary;

3. an analysis of individual constraints and variables is performed to determine whether some constraints and variables can be replaced by others.

4 PREPARATORY STAGES OF THE OPTIMIZATION CALCULATION

4.1 Identification of input data critical for the optimization calculation

Subchapter 4.1 is mainly devoted to the problem of the recommended arrangement of input data into a structure corresponding to the requirements of the optimization calculation. The arrangement of the input data will subsequently yield information about specific coordination links, which will be a direct input to mathematical models for solving network node time coordination problems.

The following types of tables can be used to organize the input data related to the optimization calculation, see Table 4.1 - Table 4.3:

Line number	Direction 1 (Departure stop → Destination stop)	Direction 2 (opposite direction if it is a shuttle line) (Departure stop → Destination stop)

Table 4.1: List of coordinated lines and their directions

<table>
<tr><th rowspan="2">Name of the transfer node</th><th rowspan="2">Transfer node number</th><th colspan="2">from</th><th colspan="2">for</th><th rowspan="2">Transfer time</th></tr>
<tr><th>line number</th><th>direction number</th><th>line number</th><th>direction number</th></tr>
</table>

Table 4.2: List of coordination nodes with coordinated lines, their directions and transfer times between their connections

The values in the last column of Table 4.2 are given in the selected time units (usually minutes).

Transfer node number	from			Volume of transferring passengers	for	
	line number	direction number	connection number		line number	direction number

Table 4.3: List of coordination links and volumes of transferring passengers using the given coordination link and transfer times between the coordinated lines

If some line numbers are omitted in the number series containing the coordinated link numbers (either the corresponding lines do not exist or are not included in the network node time coordination), it is advisable to renumber the lines so that there are no gaps between their numbers in order to simplify the optimization calculation process. For the purpose of renumbering it is advisable to prepare a transformation table, e.g. in the form shown in Table 4.4.

Original line number					
Line number for optimization calculation	1	2	3	..	n

Table 4.4: Transformation table

An analogous procedure can be followed in cases where public transport lines with different designations are included in the network nodal time coordination (in Prague, for example, metro lines with letter designations and city bus lines with numerical designations). In general, it is preferable to work with numerical labelling of coordinated lines in the optimization calculation.

4.2 Creating a coordination network

The first step in solving coordination tasks, in the sense of the presented methodology, is the creation of a coordination network. When creating a coordination network, it is possible to make very effective use of knowledge based on the mathematical discipline of graph theory. The coordination network can take the form of an edge-weighted non-oriented graph, an oriented graph (digraph) or a mixed graph (migraph).

In graph theory, there are several ways to represent graphs. Of these, the most illustrative for the purpose of creating a coordination network is the method using the so-called graph diagram, i.e. the coordination network is represented by a diagram containing vertices and edges (either non-oriented or oriented).

The vertices in the graph represent locations in the coordination network that are significant in terms of network node time coordination of connections (transfer nodes), the edges in the coordination network represent segments of the transport network corresponding to the routes of lines whose connections are coordinated at defined locations in the line network (transfer nodes). The edge weights correspond to the travel times between the points in the network where the connections of different lines are coordinated. The choice of the type of graph representing the network depends on the specific operating conditions that occur in the coordination network in terms of line management. If the travel times of the connections of the coordinated lines between two nodes are the same for both directions, the type of graph suitable for the construction of the coordination network is the non-oriented edge-weighted graph, see the situation shown in Figure 4.1.

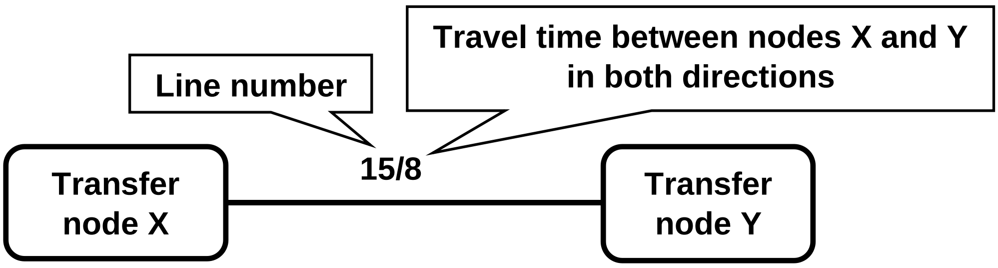

Fig. 4.1: Example of a non-oriented graph representing a fragment of a coordination network

Edge weights should be given as a pair of characters separated by a slash. The character before the slash represents the designation (usually numbers) of the lines whose routes intersect the transfer nodes. The character (number) after the slash represents the travel time given in pre-selected time units (usually minutes).

If the travel times of the coordinated lines' connections are different for each section for both directions, then the type of graph suitable for the construction of the coordination network is an oriented graph (digraph), see the situation shown in Figure 4.2.

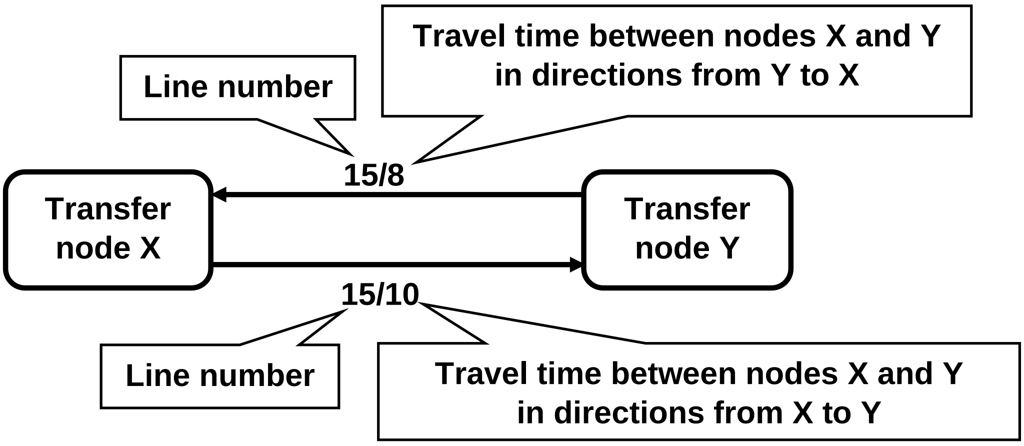

Fig. 4.2: Example of an oriented graph representing a fragment of a coordination network

If there are sections in the coordination network in which the travel times of the coordinated lines for both directions are the same, and at the same time sections in which the travel times of the coordinated lines for both directions are different, then the type of graph suitable for the construction of the coordination network is a mixed graph (migraph), see the situation shown in Figure 4.3.

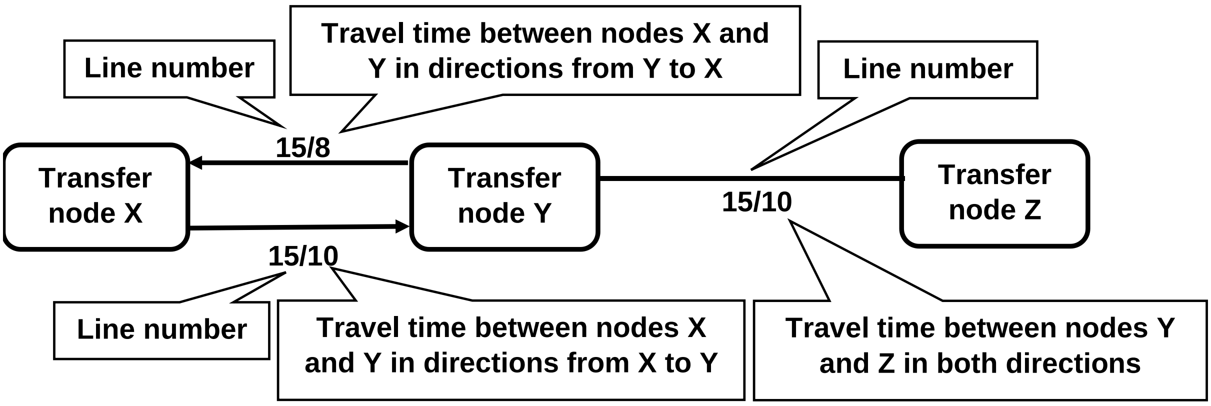

Fig. 4.3: Example of a migraph representing a fragment of a coordination network

The concept of divergence of travel times also includes situations in which sections of a given line are served in only one direction (this occurs on so-called circle lines, except in cases where there are two opposing circle lines characterised by identical travel times between network node time coordination points).

A non-oriented graph, digraph or migraph may be replaced by another type of graph, referred to as a multigraph, multidigraph or multimigraph, depending on the specific conditions. The prefix multi characterises, for example, a situation where there are several sections between two points of importance for coordination, characterised by different travel times for the same direction or even for different directions.

The basic requirement applied in the creation of the coordination network is that the coordination network should, on the one hand, contain maximum information relevant for the subsequent optimization calculation and, on the other hand, be as transparent as possible. It is therefore necessary to address the choice of a suitable description of the coordination network and the reduction of the input data to the necessary minimum.

In the first place, it is not necessary to include nodes representing final stops in the mathematical model. Their inclusion on the one hand has no effect on the result of the optimization calculation and on the other hand reduces the clarity of the network as a whole. The arrival times at the final destination stops and the departure times of the connections from the departure final stops can be derived simply by adding the values of the travel times from the transfer nodes served last in the sequence on the routes of the lines to the final destination stops, or by subtracting the values of the journey times from the departure final stops to the transfer nodes served first in sequence from the service times of the transfer nodes served first in sequence on the routes of the lines in that direction.

It is also advisable to introduce into the coordination network only lines that have at least two transfer nodes on the route with simultaneous coordination (connections of a given line are coordinated with connections of other lines in more than one node at the same time). A junction on the route of a line can therefore be considered a transfer node with ongoing coordination if it is crossed by other lines from which connections of the line in question are being transferred or to which connections of the line in question are being transferred to, while the lines in question are subject to coordination at other nodes.

In the case of lines with only one transfer node on the route, it is possible to perform network node time coordination separately from the previous (main) network node time coordination task by creating and solving a separate optimization task containing one transfer node, which will include the required coordination links, but the time shifts of connections on the coordinated lines will be enabled only in the case of a line that was not part of the previous (main) network node time coordination task. If the same transfer node is served by multiple lines that have not been included in the network node time coordination task, it is possible to include all those lines on which network node time coordination has not yet been performed in one optimization task.

5 GENERAL CONSIDERATIONS LEADING TO THE CONSTRUCTION OF A MODEL FOR NETWORK NODE TIME COORDINATION

The methodology focuses on operational situations where coordinated connections of lines are run at an headway of at least 5 minutes. The methodology is applicable even when lower headways exist, however, in these cases network node time coordination loses its relevance because the frequencies of connections of the arriving and departing lines between which passengers transfer during the coordination period are high, and thus the waiting time for transferring passengers is short enough for them to be normally acceptable.

If the transfer node on the route of the line is a transit interchange, i.e. the vehicle, after arriving at the transfer node, dropping off and picking up passengers, continues on the route of the same service on the same line, the methodology assumes that the arrival and departure time of a particular connection to/from the interchange is expressed by the same time. The methodology therefore assumes that the arrival and departure times of the same connection on the same line in the same direction are at the same time positions (which corresponds to standard public transport conditions where no differences are applied, such as in suburban or long-distance bus transport, rail transport, etc.). For this reason, the terms time of arrival of the connection to the transfer node and time of departure of the connection from the transfer node are replaced by the term time of the service at the transfer node.

The meaningfulness of solving network node time coordination problems is conditioned by the possibility of changing the time position of at least one of the connections. The means of network node time coordination is therefore to change the time positions of the connections of the coordinated lines serving the transfer nodes. In the case of headway operation, the meaningfulness of solving the tasks is conditioned by the possibility of changing the time positions of at least one of the coordinated lines. It follows from the previous sentence that it is irrelevant to include in the network node time coordination pairs of lines which are transferred between, for which there is no possibility of changing the time positions of the connections, because the current solution (the current timetable of the lines concerned) is the optimal solution.

5.1 Selection and explanation of the optimization criterion principle

In Chapter 3, the basic meaning of the optimization criterion during the optimization calculation was generally characterized. The optimization criterion is therefore a quantity whose value characterizes the quality of a particular solution of the task, while the particular solution in the case of network node time coordination tasks means the schedule of time positions of the connections of coordinated lines (i.e. the timetables of coordinated lines) conducted in headways. The objective function representing the optimization criterion must represent the legitimate interest of the submitter, which in the case of network node time coordination tasks are transport companies or public transport organizers. In the case of network node time coordination tasks, it is crucial that the legitimate interest of the submitter is as much as possible aligned with the legitimate interest of the end users of the provided service, in this case the provided service is public transport and its end users are passengers using the coordinated lines for travelling to work, education, culture, leisure activities, etc.

In order not to reduce its competitiveness, public transport must be sufficiently attractive for passengers and thus competitive with individual car transport. The attractiveness of public transport is judged by end users according to a number of criteria, in the context of the project under consideration the time criterion comes into consideration, namely the total time of transfer between sources and destinations of end users' trips.

The total transit (journey) time of a particular passenger can be formulated in general terms by the relationship (5.1) [3]:

$$t_p = t_1 + t_č + t_{dp} + t_{pre} + t_2 \tag{5.1}$$

where:

t_p the total transport time of the passenger from the start of the journey until reaching the final destination

t_1 the walking time of the passenger from the start of the journey to the first boarding stop on his/her route

$t_č$ the passenger's waiting time for the nearest connection at the first stop on the route to the final destination

t_{dp} the total time of the passenger's stay in the means of transport (journey time) on public transport routes from the start of the journey until reaching the final destination

t_{pre} the total time of passenger transfers on the route of the used public transport lines (passenger walking time and passenger waiting time for the transfer)

t_2 the passenger's walking time from the last public transport stop on his/her route to the final destination

The total transit time of all users can then be formulated in general terms using the relationship (5.2):

$$\sum_{i \in I} t_{p_i} = \sum_{i \in I} \left(t_{1_i} + t_{č_i} + t_{dp_i} + t_{pre_i} + t_{2_i} \right) \qquad (5.2)$$

where:

I the set of passengers using public transport

t_{p_i} the total transfer time of the passenger $i \in I$ from the start of the journey until reaching the final destination

t_{1_i} passenger walking time $i \in I$ from the start of the journey to the first public transport stop on the route

$t_{č_i}$ passenger waiting time $i \in I$ for the nearest connection at the first stop on the route to the final destination

t_{dp_i} the total duration of the passenger's stay $i \in I$ in the means of transport (passenger's transport time $i \in I$) on public transport routes from the start of the journey until reaching the final destination

t_{pre_i} the total time of the passenger's transfers $i \in I$ on the route of the used public transport lines (passenger walking time $i \in I$ and passenger waiting time $i \in I$ for the service at the transfer node)

t_{2_i} passenger walking time $i \in I$ from the last stop of public transport on his/her route to the final destination

Since passenger walking times from the sources of the journey to the boarding stops of public transport lines on which passengers enter the public transport system on their routes to the journey destinations, passenger walking times from the exit stops of public transport lines, on which passengers leave the public transport system on their routes to the journey destinations, and the total time spent by passengers on the means of transport (as defined by the timetable) are independent of the timing of the connections, the total transport time can only be reduced by the terms of the t_{pre} in relations (5.1) and $\sum_{i \in I} t_{pre_i}$ (5.2) representing the total transfer times of passengers. However, there is also a time component independent of the time positions of the connections and a time component dependent on the time positions of the connections. The walking times of passengers at transfer nodes (transfer time) can be considered as the time component independent of the time positions of the connections, and the waiting time of passengers for the nearest connecting connections at transfer nodes whose routes lead to the journey destination can be considered as the time dependent component. The waiting times of passengers for the nearest connecting connections at transfer nodes represent the time loss of passengers, and are therefore a factor reducing the attractiveness and therefore the competitiveness of public transport. Hence, the objective function must also represent the total time loss of all transferring passengers who transfer between connections on multiple lines during their journeys, and who wait at the transfer nodes for the arrival of the nearest connecting connections to continue to their destinations (transit). The task of the submitter, and therefore also of the researchers, is of course to reduce the total time loss of passengers changing at transfer nodes as much as possible. The aim of the optimization calculation is therefore, in the terminology of optimization methods, to minimize the value of the total time loss of all transferring passengers at all transfer nodes. In order to minimize the total time loss of all transferring passengers at all transfer nodes, it remains to answer the question of how the total time loss of all transferring passengers at all transfer nodes is to be quantified.

Transport practice shows that passenger transfers made at different transfer nodes, but also at different coordination lines at the same transfer nodes in public transport networks, are not of equal importance. It is not the same if, when changing between two lines, there is, for example, 1 passenger waiting at a particular transfer node for a connecting line and if, when changing between connections of two lines, there are, for example, 100 passengers waiting at the same time at the same transfer node. The objective function, i.e. in addition to waiting time, must also take into account the weights of transfers represented by the volume of transferring passengers. If the transfers have different importance, then also the optimization method can suggest, after the optimization calculation is completed, such time positions of the coordinated lines' connections that transfers with lower numbers of transferring passengers will cause longer waiting than transfers with higher numbers of transferring passengers. The unit of the objective function will therefore be person-minutes. The principle of time loss calculation will be demonstrated in Example 5.1.

However, since there is some variability in the number of passengers changing at junctions between different lines (i.e., the number of passengers changing at junctions between the same lines can be different on different days and in the same time period), it is desirable to choose a representative value, which can be e.g. the average number of transferring passengers per transfer, calculated as the arithmetic average of the number of transferring passengers over a selected longer period of time (e.g. week, month) per transfer, or the cumulative number of transferring passengers over that period.

Consider the following example.

Example 5.1

A pair of transfer nodes is defined in the coordination network $u = 1{,}2$ *lying on the routes of lines of certain directions, where the lines, for the sake of illustration, deviating from the recommendation made in Chapter 4, are marked e.g. with letters A, B, C. The situation is shown in detail in Fig. 5.1.*

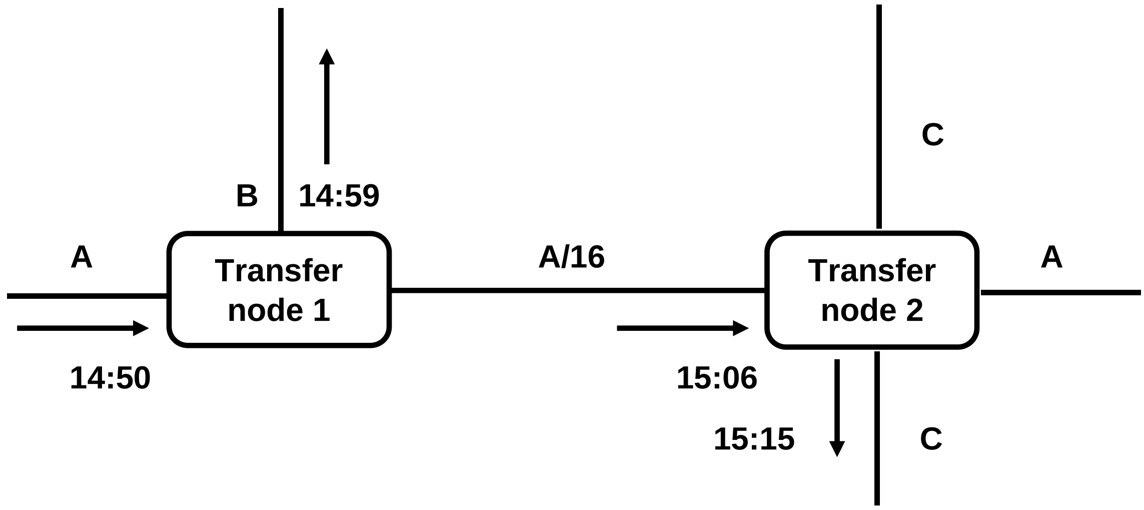

Fig. 5.1: Coordination network diagram for example 5.1

Consider one connection on each of the above lines and assume that at transfer node 1 the connection of line A transfer f_{1A} *passengers to a Line B connection in that direction and at transfer node 2 of line A in the same direction transfer* f_{2A} *passengers to a line C connection of the direction in question.*

Transfer nodes 1 and 2 are the intermediate stops of the line marked A, transfer node 1 is the departure final stop of the line marked B from the point of view of the line routing and transfer node 2 is the intermediate stop of the line marked C. The arrows represent the directions of travel of the coordinated lines.

Let $f_{1A} = 10$ *and* $f_{2A} = 5$ *be the volumes of passengers transferring from line A to line B (*f_{1A}*) at transfer node 1 and the passengers transferring from a line A connection to a line C connection at transfer node 2. Let us assume that the timetables of the three coordinated lines are designed in such a way that the line marked A serves transfer node 1 at 14:50 and transfer node 2 at 15:06. Further assume that the nearest connection of line B serves transfer node 1 at 14:59 and the nearest connection of line C serves transfer node 2 at 15:15. Let us consider that in the case of transfer node 1, the lines marked A and B do not share a common station and transfer time between the station of line A and the station of line B is 4 minutes, and in the case of transfer node 2, lines A and C share a common station, so the transfer time between the station of line A and the station of the line C is negligible (e.g. both vehicles can stop behind* *each other with a standing capacity of 2 vehicles), therefore the time needed to change between the station of line A and the station of line C is considered to be 0 minutes.*

If passengers arrive at transfer node 1 at 14:50 on a connection of the line marked A and the transfer time between the point on the line marked A and the point on the line marked B is 4 minutes, then passengers arrive at the point on the line marked B at 14:54 and their wait for the next connection departing at 14:59 is 5 minutes. However, from the perspective of transferring passengers, it would be a mistake to consider a time loss of 5 minutes regardless of the number of waiting passengers. Since the timing positions of the connections in the current timetable generate a time loss of 5 minutes for each passenger, with 10 transferring passengers and a time loss of 5 minutes for each, a total time loss of 50 person-minutes will be generated. This value is the product of the number of transferring passengers and the generated waiting time. Similarly, we calculate the total time loss for passengers who change from a line A connection to a line C connection at transfer node 2. If 5 transferring passengers arrive at transfer node 2 at 15:06 on line A and the transfer time between the station of line A and the station of line C is 0 minutes, then the time position of the line A service (15.06) is different from the time position of the line C service (15.15) is 9 minutes away and therefore for each of the transferring passengers these time positions generate a time loss of 9 person-minutes for waiting for the next connection of line C departing at 15:15, thus the total time loss of all 5 transferring passengers is 45 person-minutes. The total time loss of passengers changing at both transfer nodes is therefore 50 + 45 = 95 person-minutes.

Thus, the total time loss of transferring passengers at both transfer nodes is the sum of the total time loss generated at all transfer nodes where network node time coordination takes place.

In Chapter 2, devoted to the subject of the methodology, it was stated that in order to be meaningful in solving network node time coordination problems, at least one connection of one line must be allowed to change its temporal position. If this is not the case, the current timetables represent the optimal solution. Possible changes in the time positions of lines are reflected in network node time coordination tasks by changes in the service times of transfer nodes located on the routes of the coordinated lines. The principle of the correlation between the time shift possibilities of the coordinated lines and the value of the objective function will be explained in Example 5.2.

<u>*Example 5.2*</u>

We assume possible changes in the time positions of the line A connection in the interval $\langle -4;\ +2\rangle$ *minutes, in the case of a line B connection in the interval* $\langle -0;\ +0\rangle$ *minutes, and in the case of a line C connection at an interval of* $\langle -1;\ +2\rangle$ *minutes. This means in fact that the service time of transfer node 1 by line A can take place at 14:46 at the earliest and at 14:52 at the latest and the service time of transfer node 2 by line A can take place at 15:02 at the earliest and at 15:08 at the latest. The time of service of transfer node 1 by the line B connection may take place at 14:59 at the earliest and at 14:59 at the latest (the time position of the line B connection cannot be changed) and the time of service of transfer node 2 by the line C connection may take place at 15:14 at the earliest and at 15:17 at the latest. The waiting time for passengers transferring from line A to line B at transfer node 1 in the current state is 4 minutes. In order to reduce the total time loss of transferring passengers in the next procedure, let's change the time position of the line A connection in the permissible value of +2 minutes (this will move the service of transfer node 1 by the line A connection to the latest time position) and observe how the value of the total time loss of all transferring passengers at both transfer nodes changes. By moving the line A connection to the outermost time position we get the service of transfer node 1 at 14:52 and the service of transfer node 2 at 15:08. The value of the total time loss of all passengers changing at transfer node 1 will now be 30 person-minutes and the value of the total time loss of all passengers changing at transfer node 2 will now be 35 person-minutes, i.e. the total time loss of all passengers changing at both transfer nodes will now be 65 person-minutes. Since the original value of the total time loss of all transferring passengers in both transfer nodes was 95 person-minutes, the implementation of the change of the time position of the line A connection resulted in a reduction of the value of the objective function by 30 person-minutes, so we can see that the network node time coordination* *in transfer node 1 brought a reduction in the total time loss of transferring passengers, and therefore also an increase in the attractiveness and consequently the time competitiveness of the coordinated public transport connections. Let us further note that in the case of transfer node 1, the implementation of the time shift of the line A connection reduced the total time loss from 50 person-minutes to 30 person-minutes and in the case of transfer node 2, the implementation of the time shift of the line A connection reduced the total time loss from 45 person-minutes to 35 person-minutes. That is, changing the time position of the service by the same value at both transfer nodes resulted in different values of partial time loss reduction. In the case of transfer node 1, there was a more significant reduction in the total time loss than in the case of transfer node 2, which corresponds to the general consideration that the number of transferring passengers also influences the amount of time loss. While the absolute value of the time loss is reduced by 2 minutes in both cases (because the line A connection is delayed by +2 minutes, which is reflected in both transfer nodes equally), the value of the reduction of the total time loss at transfer node 1 is 2 times higher than the value of the reduction of the total time loss at transfer node 2, because twice as many passengers change at transfer node 1 as at transfer node 2 and therefore, for every minute of time loss saved, the total time loss at transfer node 1 is reduced by twice the time loss at transfer node 2.*

It remains to be decided whether the implementation of the time shift of line A of +2 minutes has achieved the time positions of the connections that actually generate the minimum value of the total time loss at both transfer nodes. In order to achieve the minimum value of the total time loss at both transfer nodes, it must not be the case that the time positions of the coordinated connections could be brought even closer together. The above demonstration example is so illustrative and simple that even without the use of an optimization calculation, it is relatively easy to see that the time positions generating the minimum value of the total time loss at both transfer nodes have still not been reached. There is another way to bring the time positions of the connections closer to each other. The reduction of the total time loss at transfer node 1 can no longer be achieved because the line B connection, which is transferred from line A at transfer node 1, cannot change its time position so that the time positions of *the two connections of lines A and B are brought closer together. However, a reduction in the value of the total time loss can be achieved at transfer node 2. From the interval* $\langle -1; +2 \rangle$ *minutes defining the possible changes of the time position of the line C connection, it follows that the time of service of the transfer node 2 by the connection of the given line can be moved backward by 1 minute, i.e. it can be moved from the time position 15:15 to the time position 15:14. Considering the number of transferring passengers in transfer node 2, which is 5, the moving of the service time of transfer node 2 by the line C connection by 1 minute will result in a reduction of the total time loss in transfer node 2 by another 5 person-minutes, i.e. the total time loss of transferring passengers in both transfer nodes will reach a minimum value of 60 person-minutes after both implemented changes (time shift of the line A connection by +2 minutes and time shift of the line C connection by -1 minute).*

The objective function has a summation form. In this context, however, it is important to point out one of the pitfalls of this principle. This, paradoxically, is related to the importance of coordination links mentioned above. Taking these into account, it is possible that some coordination links with significant importance (large volumes of transferring passengers) may be preferred over coordination links with low importance to such an extent that the total time loss for coordination links with low importance may exceed the maximum acceptable value for transferring passengers, which may even cause the termination of the coordination link due to loss of passenger interest in public transport. In such a case, the optimization approach must be supplemented with additional limiting factors to prevent the unacceptable value of time loss from occurring. However, if such a constraint is included in the optimization model, it should be kept in mind that the inclusion of this type of constraint may cause an undesirable deviation from the optimal solution in terms of the total time loss (increase in the value of the total time loss).

As already mentioned in general terms in Chapter 3, the requirement for the type of extreme sought is also an integral part of the objective function. From the above examples and the nature of the optimized quantity, it follows that the efficient type of the search variable extreme, which is the total time loss of transferring passengers at all transfer nodes, will be the minimum.

If passengers at the transfer node change to a one-way service, the situation is simple, as there is no need to differentiate the direction. In case passengers change to more than one direction at the transfer node, it is necessary to differentiate the directions. If there are transfer nodes in the network where passengers change to more than one direction and parallel transfer nodes where passengers change to one direction, it is desirable to maintain a unified approach in terms of model building, i.e. to consider the direction even in the case of a transfer node where passengers change to only one direction.

5.2 Limiting factors

As already mentioned in Chapter 3, limiting factors define the set of admissible solutions in the problem at hand. The limiting factors are represented in terms of mathematical models by so-called constraints.

The admissible solution in the problem is characterized by the following limiting factors.

In the first place, coordination links must not be made from connections of incoming lines to connections whose departure times from the transfer node according to the timetable are earlier than the arrival times of the connections according to the timetable from which the transfer is being made. Another limiting factor is to allow time shifts of connections that suit the intervals at which the time shifts can be implemented. In the case of minor coordination links with unacceptable values of time loss of transferring passengers, constraints must be included to ensure that the maximum acceptable value of time loss is not exceeded.

Further, the limiting factors are the domains of definition of the variables that represent decisions in the proposed models or create logical links between the values of the variables. An essential part of the system of constraints will also be the binding constraints, which will create a link between the system of constraints and the objective function whose value is minimized in the optimization calculation.

6 METHODOLOGY FOR CREATING THE OPTIMIZATION MODEL

When coordinating connections at transfer nodes using the approaches described in this methodology, and the models were based on [4], it is necessary to take effective care to minimize the size of the optimization model. As the size of the optimization model grows, the computational complexity also grows, i.e. the demands on the use of the PC operating memory increase and the computation time increases. If the size of the optimization model exceeds a certain size, the situation may also arise that the optimal solution cannot be achieved, taking into account the capacity capabilities of the computing technology.

The size of the optimization model is generally assessed by the number of structural constraints and the variables used (especially integer variables). In coordination problems, the number of constraints and the number of integer (in this case, bivalent) variables depend on the number of connections, while the number of structural constraints and the number of binary variables also increase with the number of connections included in the optimization calculation. In the conditions of headway operation the solver has simplified the situation by the fact that it is possible to introduce the same variable for modelling the shifts of all connections of a specific line running in a specific direction during the coordination period.

Before starting the process of entering input data into the optimization model, it is necessary to solve some problems related to time and other aspects affecting the course of the optimization calculation. These aspects affecting the course of the optimization calculation are:

1. setting the default time positions of the connections before starting the optimization calculation (see subsection 6.1),
2. time modelling in the mathematical model (see subsection 6.2),
3. defining the values of maximum time shifts of connections of coordinated lines in the coordination period (see sub-chapter 6.3),

4. identification of the necessary minimum number of connections of coordinated lines (see subchapter 6.4).

6.1 Setting the default time positions of the connections before starting the optimization calculation

The first problem is caused by the simultaneous existence of a possible hastening and delay of the service time of a transfer node by a particular connection. Since in mixed integer linear programming it is generally not possible to work with negative values of variables (variables have domains of definition R_0^+, Z_0^+ and $\{0; 1\}$), it is problematic to use the same variable representing the time delay of the specified connection to model the haste and also the delay at the transfer node by the specified connection.

The problem can be solved, for example, by introducing a pair of variables for each specific connection, one of which will model the shift of the connection from the current time position to a later time position and the other of which will model the shift of the connection from the current time position to an earlier time position. However, the introduction of separate variables modelling the time shifts of connections from the current time positions to later time positions and separate variables modelling the time shifts of connections to earlier time positions within predefined intervals is unnecessarily complicated, because the number of variables in the mathematical model would be redundant (if we consider the introduction of a pair of variables for all connections of coordinated lines in specific directions, regardless of whether or not shifts in those directions are possible).

From the point of view of efficiency of calculation and clarity of the solution, it is therefore much more suitable to implement the second method, which consists in moving all coordinated connections to their extreme time positions before the start of the optimization process (the extreme time positions correspond to the earliest possible or latest permissible time positions of the connections), which allows to use only one variable for the time shift of connections, because the time shifts of the coordinated lines will take place only in one direction. The principle of the mathematical model does not prevent the shifts of the connections to the earliest possible or latest permissible time positions to be combined with each other, however, from the point of view of practical solution and clarity, this combination of variables cannot be recommended. In the following procedure, the text of the methodology will consider the option of moving connections to the earliest possible time positions.

However, when implementing the shifts of connections to their extreme time positions (valid for both extreme time positions) before the start of the optimization process, it is necessary to keep in mind that the actual time shift of the connection may differ from the calculated time shift of the connection. If, for example, the value of the permissible time delay of a specific connection is defined by a closed interval of $\langle -3; +5 \rangle$ minutes, the connection is moved to the earliest possible time position before the start of the optimization calculation, and if a time shift of +3 minutes is proposed after the optimization calculation is completed, then the proposed time position of the connection corresponds to the current time position of the service, and thus the current time position of the connection is not actually changed.

6.2 Modelling time in a mathematical model

The second problem is the modelling of time for the requirements of the optimization software. As a rule, universal optimization software does not work with time values in real format, i.e. hh:mm, for optimization calculations. To this end, it is appropriate to introduce an internal calendar for the coordination period. The internal calendar is a commonly used term, which is used for example in network analysis.

In the conditions of the solved tasks, the introduction of the internal calendar means the selection of a specific time point, where the selected time point is assigned the value 0, from which the time is calculated in the selected time units. The value 0 can also represent e.g. midnight of a given day or any other arbitrary point in time. In the conditions of the solved tasks, the value 0 will represent the beginning of the coordination period. The chosen time unit can be either an hour or a minute, in the framework of the presented methodology the chosen time unit will be 1 minute, which is not only advantageous in terms of other time variables used in the model, e.g. interval values, which are commonly given in minutes, but it is also common from the point of view of the theoretical considerations made in subsection 5.1.

6.3 Defining values of maximum time shifts of connections of coordinated lines in the coordination period

The next step in the preparation of the mathematical model is the identification of time values to which it will be possible to implement time shifts of the connections of the coordinated lines. From the point of view of the optimization calculation it is essential to identify the values of maximum time shifts of the connections of coordinated lines. The key parameters for identifying the values of maximum time shifts of connections of coordinated lines are:

1. the headway values between the connections of coordinated lines applied during the coordination period,

2. values of elementary time units in terms of time coordination on coordinated lines.

The elementary time units may differ on coordinated lines in terms of time coordination, e.g. in the case of tram, trolleybus and bus lines, the elementary time unit of 1 minute is usually applied in terms of time coordination, in the case of the metro, the elementary time unit of e.g. 30 seconds is applied in terms of time coordination.

In the context of the applied principle of determining the initial time positions of the connections of coordinated lines before starting the optimization calculation (i.e. moving the connections to the extreme positions corresponding to the earliest possible times serving the transfer nodes), the values of the maximum time shifts will correspond to the earliest possible time positions within the defined values of the headways. Maximum value of the time delay of connections on a coordinated line $i \in L$ (where L represents the set of coordinated lines) can be calculated from the relationship (6.1):

$$T_i - e_i \tag{6.1}$$

where:

T_i.. the value of the constant headway or the value of the alternating headway applied between the connections of the line $i \in L_u$

e_i.... elementary time unit in terms of time coordination on a coordinated line $i \in L$

Using the values of the constant headway and the elementary time unit in terms of time coordination applied on the line $i \in L$, it is then also possible to calculate the number of connection time position variations usable within a single interval, from relationship (6.2):

$$\frac{T_i}{e_i} \tag{6.2}$$

As the value of the elementary time unit shortens in terms of time coordination, the number of connection time positions variants on the coordinated line increases, allowing time shifts of the connections, while maintaining the interval value.

6.4 Identification of the necessary minimum number of coordinated connections of coordinated lines

From the point of view of solvability of the mathematical model of network node time coordination, it is important that the size of the model in terms of the number of variables and the number of constraints is not unnecessarily large, because the larger the mathematical model, the more likely one can expect computational complications after starting the optimization calculation. Since the size of the model depends mainly on the number of connections that are subject to time coordination, it is reasonable to limit the number of connections to be included in the network node time coordination model to the minimum necessary.

The necessary number of connections included in the coordination is influenced by the following factors:

1. the value of the coordination period length,

2. the values of the headway rates applied on the coordinated lines,

3. the type of headways between adjacent connections applied on coordinated lines,

4. values of elementary time units in terms of time coordination applied on coordinated lines.

In any case, care must be taken that passengers must be able to transfer (create coordination links for all passengers) from all connections of arriving lines to the transfer nodes where network node time coordination takes place.

The necessary number of connections included in the coordination corresponds to the number of connections whose time positions are part of the coordination period set for the network as a whole. Therefore, when identifying the necessary number of connections to be included in the coordination, it is first necessary to calculate the value of the coordination period set for the coordinated network as a whole.

6.4.1 Calculation of the coordination period value for the coordinated network as a whole

The calculation of the value of the coordination period for the coordinated network as a whole is preceded by the calculation of the value of the so-called node coordination periods. The node coordination period is calculated separately for each node in which network node time coordination takes place.

<u>Calculation of the node coordination period value</u>

The calculation of the value of the node coordination period depends mainly on the types of headway that are applied on the individual lines coordinated at the individual transfer nodes.

The simplest case is an operational situation where constant headways with the same values are applied between the connections of all coordinated lines. That is, when a headway of e.g. 20 minutes is applied between the connections of all coordinated lines between which coordination links are to be provided. In this case, the value of the node coordination period is equal to the value of the headway between the connections of all coordinated lines, i.e. 20 minutes.

In the case of an operational situation where different values of constant headway apply between the connections of coordinated lines, the value of the node coordination period is calculated as the smallest common multiple of the headways between the connections on the coordinated lines serving the transfer node. The solution procedure is demonstrated in example 6.1

Example 6.1

The transfer node is served by connections of two coordinated lines with different headways of constant lengths, namely 6 minutes and 8 minutes. The smallest common multiple of 6 and 8 is 24. The node coordination period will therefore be 24 minutes.

In the case of an operational situation where a constant-length headway is applied between the connections of some coordinated lines serving a transfer node and an alternating headway is applied in parallel on some lines, the length of the node coordination period must be calculated as the smallest common multiple of the values of all constant-length headways serving the transfer node and the values of the partial coordination periods applied on all alternating headways of lines serving the transfer node. The solution procedure is demonstrated in example 6.2.

Example 6.2

The transfer node is served by connections of two coordinated lines, one of the coordinated lines has a 6 minute headway between connections and the other of the coordinated lines has an alternating headway with periodic repeating values of 7 and 8 minutes (no matter in which order). We start the calculation of the node coordination period value by calculating the partial coordination period for the line on which the alternating headway is applied. The value of the partial coordination period for a line with an alternating headway is (according to the definition of the partial coordination period for a line given in Chapter 1) 15 minutes (this is the sum of the individual alternating headway values). Then we calculate the smallest common multiple of the value of the constant headway of 6 minutes and the value of the partial coordination period of 15 minutes. The value of the node coordination period is 30 minutes.

In the case of an operational situation where an alternating headway is applied between the connections of all coordinated lines serving a transfer node, the value of the node coordination period must be calculated as the smallest common multiple of the partial coordination periods for the lines serving the transfer node, see Example 6.3.

Example 6.3

The transfer node is served by connections of two coordinated lines, with alternating headways between the connections on both coordinated lines, one of the coordinated lines with periodic repeating values of 30, 20 and 10 minutes (in whatever order) and the other of the coordinated lines with periodic repeating values of 7 and 8 minutes (in whatever order). We start the calculation of the node coordination period value by calculating the partial coordination periods for individual lines. The value of the partial coordination period for a line with periodic repeating values of 30, 20 and 10 minutes is 60 minutes, the value of the partial coordination period for a line with periodic repeating values of 7 and 8 minutes is 15 minutes. Then we calculate the least common *multiple of the values of the partial coordination periods for both lines of 60 minutes and 15 minutes, the value of the least common multiple is 60 minutes and therefore also the value of the node coordination period is 60 minutes.*

Calculation of the coordination period value for the coordinated network as a whole

The coordination period for the coordinated network as a whole will be the smallest common multiple of the node coordination period values calculated for all transfer nodes in the coordinated network. To illustrate the procedure for calculating the value of the coordination period for the coordinated network as a whole, we show the case of a network containing a pair of transfer nodes connected by the route of a line that is also subject to coordination, see Example 6.4.

Example 6.4

Consider the case in which two transfer nodes are defined in a coordinated network. One of the transfer nodes has a defined node coordination period of 42 minutes and the other of the transfer nodes has a defined node coordination period of 56 minutes. The task is to calculate the value of the coordination period for the coordinated network as a whole. The smallest common multiple of 42 and 56 is 336 and thus the value of the coordination period for the coordinated network as a whole is 336 minutes.

The values of coordination periods for the coordinated network as a whole are calculated analogically in case of the occurrence of alternating headways, i.e. as the smallest common multiple of the values of node coordination periods calculated using the partial coordination periods for lines (with the occurrence of alternating headways).

6.4.2 Calculation of the minimum number of connections on arrival and departure to/from the transfer node in the coordination period

The second step leading to the creation of the optimization model is the calculation of the number of connections of the coordinated lines in the coordination period. When calculating the number of connections included in the coordination of lines at the transfer node during the coordination period, two main principles must be observed:

1. the number of connections included in the coordination period must characterise the operation over the whole coordination period,
2. for each incoming connection subject to coordination, a transfer to at least one outgoing connection within the required coordination link must be provided.

Principle 2 states that it must be possible to change from any incoming connection to at least one outgoing connection during the coordination period. Compliance with Principle 2 must be understood as a basic condition for the functionality of the mathematical model. If it is necessary to provide the possibility to change from any incoming connection, this shall also be possible for the last connection of the incoming line included in the coordination period located at the latest possible time position after any time shift.

The number of connections on the lines included in the coordination period shall be calculated separately for each of the lines subject to coordination. The basis for calculating the number of connections to be coordinated is the length of the coordination period K_p and the value of the headway T_i between connections on a given line $i \in L$.

When calculating the number of connections, however, it is still necessary to distinguish two basic cases:

1. on lines between which the coordination of connections at the transfer node takes place, a constant headway is introduced,

2. an alternating headway is introduced on the lines between which the coordination of connections at the transfer node takes place.

Calculation of the number of connections included in the coordination in case of constant headway occurrences

Consider a transfer node $u \in U$ in which network node time coordination between the arriving line $i \in L_u$ and the departing line $j \in L_u$ must be ensured.

The number of connections of the incoming line $i \in L_u$ included in the coordination period at the transfer node $u \in U$, on which the connections are routed in a constant headway, is calculated from relationship (6.3):

$$n_{ui} = \frac{K_p}{T_i} \tag{6.3}$$

where:

n_{ui} number of line connections $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period

K_p length of the coordination period

T_i the value of the constant headway applied on the incoming line $i \in L_u$

Then we calculate the latest time position of the last connection of the arriving line in the coordination period from the relationship (6.4):

$$t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i \tag{6.4}$$

where:

t_{ui1} the earliest possible service time of the transfer node $u \in U$ by the first connection of the incoming line $i \in L_u$ in the coordination period

n_{ui} number of line connections $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period

T_i the value of the constant headway applied on the incoming line $i \in L_u$

a_i value of the maximum permissible time shift of connections on a incoming line $i \in L$

For the latest time position of the last connection of the departing line $j \in L_u$ the following relationship (6.5) must be vaild:

$$
\begin{gathered}
t_{uj1} + \left[\left(n_{uj} - 1\right) - 1\right] \cdot T_j + a_j < \\
< t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i + tprest_{uij} \leq \\
\leq t_{uj1} + \left(n_{uj} - 1\right) \cdot T_j + a_j
\end{gathered}
\tag{6.5}
$$

where:

- t_{ui1} the earliest possible service time of the transfer node $u \in U$ by the first connection of the incoming line $i \in L_u$ in the coordination period
- t_{uj1} the earliest possible service time of the transfer node $u \in U$ the first connection of the departing line $j \in L_u$ in the coordination period
- n_{ui} number of connections of incoming line $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period
- n_{uj} number of connections of departing line $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period
- T_i the value of the constant headway applied between the connections of the incoming line $i \in L_u$
- T_j the value of the constant headway applied between the connections of the departing line $i \in L_u$
- a_i value of the maximum permissible time shift of connections on a incoming line $i \in L$
- a_j value of the maximum permissible time shift of connections on a departing line $i \in L$
- $tprest_{uij}$ the value of the passenger transfer time between the stations of the incoming line $i \in L_u$ and the point of departure of the departing line $j \in L_u$ at the transfer node $u \in U$

The previous relationship can also be written in the following form (6.6) after modification:

$$t_{uj1} + (n_{uj} - 2) \cdot T_j + a_j < t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i + tprest_{uij} \leq \\ \leq t_{uj1} + (n_{uj} - 1) \cdot T_j + a_j \tag{6.6}$$

The control can be set

$$t_{uj1} + (n_{uj} - 1) \cdot T_j = t_{ujn_j}$$

where:

t_{ujn_j} the earliest possible service time of the transfer node $u \in U$ by the last connection of the departing line $j \in L_u$ in the coordination period,

then for the minimum number of connections of the outgoing line $j \in L_u$ included in the coordination, the following relationship (6.7) is valid:

$$n_{uj} = \frac{t_{ujn_j} - t_{uj1}}{T_j} + 1 \tag{6.7}$$

The expression $\frac{t_{ujn_j} - t_{uj1}}{T_j}$ represents the number of constant headways that occur within the coordination period between the connections of the outgoing line, and the number of connections where the number of headways is always higher by 1.

The practical procedure for calculating the minimum number of connections on coordinated lines serving transfer nodes will be demonstrated in the following three examples relating to a general transfer node $u \in U$. In the first example (Example 6.5), the calculation procedure will be demonstrated in detail in the case of a pair of lines forming a set L_u with the requirement to create one coordination link (from the connections of the line $i \in L_u$ to the connections of the line $j \in L_u$). In the second example (Example 6.6), the characteristics of the computational procedure will be reduced to the minimum necessary, but it will be demonstrated how to proceed in the case of multiple pairs of lines forming a set L_u with different operating conditions in terms of the headways used. The third example (Example 6.7) demonstrates the procedure for calculating the minimum number of connections in the case of three lines, where it is necessary to create coordination links between the connections of two pairs of lines.

<u>Example 6.5</u>

At the transfer node $u \in U$ served by a set of lines L_u there is a time coordination of the connections of two lines $i \in L_u$ and $j \in L_u$, where $j \neq i$. First, the possible time position of the first connection of the line $i \in L_u$ is at time 3, the earliest possible time position of the first connection of the line $j \in L_u$ is at time 2, i.e $t_{ui1} = 3$ and $t_{uj1} = 2$. Connections on the incoming line $i \in L_u$ are run at a constant headway of 14 minutes, connections on the outgoing line $j \in L_u$ are run at a constant headway of 6 minutes, i.e $T_i = 14$ minutes and $T_j = 6$ minutes. Assume the value of the transfer time between the stations at which the coordinated lines stop is $tprest_{uij} = 5$ minutes and the elementary time unit in terms of time coordination for both coordinated lines is 1 minute. Furthermore, assume that for the possible time shifts of connections we use by default all permissible time positions of connections within the headway values (permissible time positions are all integer values of time shifts of connections in the headway representing the possible time shift). The task is to calculate the minimum number of connections on the lines included in the coordination at the transfer node $u \in U$.

We start the solution of the example by calculating the values of the maximum allowed time delays of the line's connections $i \in L_u$ and $j \in L_u$. Since in the case of both lines we use all allowed time positions of the connections within the headways, then, in the case of the line $i \in L_u$ it is $a_i = 13$ minutes and in the case of the line $j \in L_u$ it is $a_j = 5$ minutes.

The solution continues by calculating the value of the coordination period. Recall that we calculate the value of the coordination period as the smallest common multiple of the headways between the connections of the coordinated lines. The smallest common multiple of the numbers 14 and 6 is the number 42, so the value of the coordination period will be $K_p = 42$ minutes.

In the next procedure we calculate the number of connections of the incoming line $i \in L_u$ included in the coordination at the transfer node $u \in U$, see the relationship (6.3).

$$n_{ui} = \frac{K_p}{T_i} = \frac{42}{14} = 3$$

Then we calculate the latest time position of the last connection of the arriving line in the coordination period to the transfer node $u \in U$, from the relationship (6.4):

$$t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i = 3 + (3 - 1) \cdot 14 + 13 = 44$$

and the time position when passengers from the last connection of the line serving the transfer node arriving to the transfer node $u \in U$ at the latest time position in the coordination period are ready for the departure of the outgoing connection, from the relationship:

$$t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i + tprest_{uij} = 44 + 5 = 49$$

Then we calculate the time position of the latest connection of the outgoing line from the transfer node $u \in U$, from the relationship (6.5):

$$t_{uj1} + \left(n_{uj} - 2\right) \cdot T_j + a_j < t_{ui1} + (n_i - 1) \cdot T_i + a_i + tprest_{uij} \leq$$

$$\leq t_{uj1} + \left(n_{uj} - 1\right) \cdot T_j + a_j$$

So we are looking for the value of n_{uj}, for which

$$2 + \left(n_{uj} - 2\right) \cdot 6 + 5 < 49 \leq 2 + \left(n_{uj} - 1\right) \cdot 6 + 5$$

the minimum number of connections of the outgoing line n_{uj} is the number that is the solution of the inequality:

$$49 \leq 2 + \left(n_{uj} - 1\right) \cdot 6 + 5$$

and at the same time solving the inequality:

$$2 + \left(n_{uj} - 2\right) \cdot 6 + 5 < 49$$

The intersection of the intervals that are solutions of both inequalities is the interval

$$8 \leq n_{uj} < 9$$

and the minimum number of connections of the outgoing line corresponds to an integer value lying in the given interval. Thus, the minimum number of connections of the outgoing line included in the time coordination within the coordination period is $n_{uj} = 8$*. The coordination period therefore contains 8 connections of the outgoing line.*

Example 6.6

At the transfer node $u \in U$ *served by a set of lines* L_u *there is a time coordination of the connections of two lines* $i \in L_u$ *and* $j \in L_u$*, where* $j \neq i$*. First, the possible time position of the first connection of line* $i \in L_u$ *is at time 1, the earliest possible time position of the first connection of line* $j \in L_u$ *is at time 3, i.e* $t_{ui1} = 1$ *and* $t_{uj1} = 3$*. Connections on line* $i \in L_u$ *are run at a constant headway of 10 minutes, connections on line* $j \in L_u$ *are run at a constant headway of 6.5 minutes, i.e* $T_i = 10$ *minutes and* $T_j = 6{,}5$ *minutes. Assume the value of the transfer time between the stations at which the connection of the coordinated lines stop* $tprest_{uij} = 2$ *minutes and the elementary time units in terms of time coordination are 1 minute for the arriving line and 0.5 minutes for the departing line. Further, by analogy with the previous example, assume that for the possible time shifts of connections we use by default all permissible time positions of connections within the headway values for both lines (permissible time positions are integer values of time shifts of connections for the incoming line, and both integer values of time positions of connections and half-minute values of time positions of connections for the outgoing line). The task is to calculate the minimum number of connections on the lines included in the coordination at the transfer node* $u \in U$*.*

To solve the example, we will again start by calculating the values of the maximum allowed time delays of line connections $i \in L_u$ *and* $j \in L_u$*. In the case of line* $i \in L_u$ *it is* $a_i = 9$ *minutes, in the case of line* $j \in L_u$ *it is* $a_j = 6$ *minutes.*

The solution continues by calculating the value of the coordination period. The smallest common multiple of 10 and 6.5 is 130, so the value of the coordination period will be $K_p = 130$ *minutes.*

In the next procedure we calculate the number of connections of the incoming line $i \in L_u$ *included in the coordination at the transfer node* $u \in U$*, see relation (6.3).*

$$n_{ui} = \frac{K_p}{T_i} = \frac{130}{10} = 13$$

Then we calculate the latest time position of the last connection of the arriving line in the coordination period from the relationship (6.4):

$$t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i = 1 + (13 - 1) \cdot 10 + 9 = 130$$

and the time position when passengers from the last connection of the arriving line serving the transfer node $u \in U$ *at the latest time position in the coordination period reach the point of departure of the departing line connection from the relationship:*

$$t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i + tprest_{uij} = 130 + 2 = 132$$

We then calculate the time position of the latest connection of the outgoing line from the relationship (6.5):

$$3 + \left(n_{uj} - 2\right) \cdot 6{,}5 + 6 < 132 \leq 3 + \left(n_{uj} - 1\right) \cdot 6{,}5 + 6$$

The intersection of the intervals that are solutions of the corresponding inequalities is the interval

$$19{,}92 \leq n_{uj} < 20{,}92$$

i.e., the minimum number of connections of the outgoing line included in the time coordination within the coordination period is $n_{uj} = 20$*. The coordination period therefore contains 20 connections of the outgoing line. That this is the correct solution can be ascertained by simple reasoning based on a comparison of the latest value of the time of arrival of passengers from the last connection of the arriving line serving the transfer node to the stop of the departing line serving the transfer node and the* *latest value of the departure of the last connection of the departing line serving the transfer node. The value of the latest time of arrival of passengers from the last connection of the arriving line serving the transfer node to the stop of the departing line serving the transfer node is 132 and the latest value of departure of the last connection of the departing line serving the transfer node is 133, thus the transfer from the last connection of the arriving line at the latest time position to the last connection of the departing line at the latest time position is assured.*

Example 6.7

At the transfer node $u \in U$ *served by a set of lines* L_u *there is a time coordination of the connections of two pairs of lines, namely a pair of arriving lines* $i \in L_u$ *and the outgoing line* $j \in L_u$*, where* $j \neq i$ *and the incoming line pair* $k \in L_u$ *and the outgoing line pair* $j \in L_u$*, where* $k \neq i$ *and simultaneously* $k \neq j$*.*

Input information for the pair of lines $i \in L_u$ *and* $j \in L_u$*, where* $j \neq i$*, are: earliest possible time position of the first connection of the line* $i \in L_u$ *is at time 3, the earliest possible time position of the first connection of the line* $j \in L_u$ *is at time 2, i.e* $t_{ui1} = 3$ *and* $t_{uj1} = 2$*. Connections on the line* $i \in L_u$ *are run at a constant headway of 14 minutes, connections on the line* $j \in L_u$ *are run at a constant headway of 6 minutes, i.e* $T_i = 14$ *minutes and* $T_j = 6$ *minutes. We assume the value of the transfer time between the stations at which the connections of the coordinated lines stop* $i \in L_u$ *and* $j \in L_u$*, where* $j \neq i$*,* $tprest_{uij} = 5$ *minutes and the elementary time units in terms of time coordination for both lines correspond to 1 minute. Further, by analogy with the previous two examples, we assume that for the possible time shifts of the connections we use by default all permissible time positions of the connections within the headway values (the permissible time positions of the connections are thus integer values of the time shifts of the connections).*

Input information for the third of the lines, the line $k \in L_u$*where* $k \neq i$ *and at the same time* $k \neq j$*, are as follows. First, the possible time position of the first connection of the* *line* $k \in L_u$ *is at time 1.5, i.e* $t_{uk1} = 1{,}5$. *Connections on the line* $k \in L_u$ *are run at a constant 20-minute headway, i.e* $T_k = 20$ *minutes. We assume the value of the transfer time between the stations at which the coordinated lines stop* $k \in L_u$ *and* $j \in L_u$, *where* $k \neq i$ *and at the same time* $k \neq j$, $tprest_{ukj} = 3$ *minutes and an elementary time unit in terms of time coordination of 0.5 minutes. Furthermore, we assume that for possible time shifts of the line connections* $k \in L_u$*where* $k \neq i$ *and at the same time* $k \neq j$, *we again use by default all the allowable time positions of the connections within the interval values (the allowable time positions are in the case of line* $k \in L_u$ *in addition to the integer values of the connection time shifts, also the half-minute values of the connection time shifts).*

The task is to calculate the minimum number of connections on all three lines involved in the coordination at the transfer node $u \in U$.

The values of the maximum permissible time delays of connections are in the case of line $i \in L_u$ $a_i = 13$ *minutes and for line* $j \in L_u$ $a_j = 5$ *minutes. In the case of line* $k \in L_u$*where* $k \neq i$ *and at the same time* $k \neq j$, *the value of the maximum allowed connection time delay is* $a_k = 19{,}5$ *minutes.*

In the case of three lines, two of which are incoming and one of which is outgoing, it is necessary to proceed differently than in the examples discussed so far. The value of the coordination period will be set as the smallest common multiple of the headway on all three lines and the last connection of the outbound line at the latest time position must ensure a transfer from the last connections of both inbound lines serving the transfer node at the latest time positions (including transfer times). Coordination period value K_p *is the smallest common multiple of 14, 6 and 20 minutes, so the value of the coordination period is* $K_p = 420$ *minutes.*

The minimum number of connections of the incoming line $i \in L_u$ *included in the coordination is* $n_i = 30$, *the latest time position of the last connection of the arriving line in the coordination period is* 422 *and the time position at which passengers from the last connection of the arriving line serving the interchange at the latest time position in the coordination period reach the stop of the departing line is* 427.

The minimum number of connections of the incoming line $k \in L_u$ included in the coordination is $n_k = 21$, the latest time position of the last connection of the arriving line in the coordination period is 421 and the time position at which passengers from the last connection of the arriving line serving the transfer node at the latest time position in the coordination period reach the stop of the departing line is 424. Since the transfer must be ensured from the last connections of both arriving lines serving the transfer node at the latest time positions (including transfer times), it is necessary to select the maximum of the two calculated values for the calculation of the minimum number of connections and insert it into the middle part of the inequality (6.5), which in example 6.7 will have the form (6.8):

$$\begin{gathered} t_{uj1} + \left[\left(n_{uj} - 1\right) - 1\right] \cdot T_j + a_j < \\ < max\left\{t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i + tprest_{uij}; t_{uk1} + (n_{uk} - 1) \cdot T_k \right. \\ \left. + a_k + tprest_{ukj}\right\} \leq \\ \leq t_{uj1} + \left(n_{uj} - 1\right) \cdot T_j + a_j \end{gathered} \tag{6.8}$$

The minimum number of connections of the outgoing line included in the time coordination within the coordination period $n_{uj} = 71$. Coordination period for line transfers $i \in L_u$ therefore contains 71 connections of the outgoing line.

The relationship (6.8) can be generalized to the form (6.9), where $|L_{uj}| \geq 2$:

$$\begin{gathered} t_{uj1} + \left[\left(n_{uj} - 1\right) - 1\right] \cdot T_j + a_j < \\ < \max_{i \in L_{uj}}\left\{t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i + tprest_{uij}\right\} \leq \\ \leq t_{uj1} + \left(n_{uj} - 1\right) \cdot T_j + a_j \end{gathered} \tag{6.9}$$

where:

n_{uj} number of connections of departing line $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period

L_{uj} the set of incoming lines to the transfer node $u \in U$ from which passengers transfer to the outgoing line $j \in L_u$

It should also be noted that in the previous three examples the directions of the coordinated lines were not considered. If it is necessary to differentiate the directions of the connections of coordinated lines during coordination, it is possible to add direction indices to the designation of the individual variables used during the calculation.

If there are multiple outgoing lines from a transfer node $u \in U$ the values of the number of services of the outgoing lines serving the transfer node $u \in U$ and included in the coordination period shall be calculated separately for each departing line.

Calculation of the number of connections included in the coordination in case of alternating headways

We first consider the case where an alternating headway is applied between the connections of the incoming line and a constant headway is applied between the connections of the outgoing line.

The number of connections of the incoming line $i \in L_u$, on which the connections are routed in alternate headways, is calculated from the relationship (6.10):

$$n_{ui} = \frac{N \cdot K_p}{T_i^p} \tag{6.10}$$

where:

K_p length of the coordination period,

T_i^p value of the partial coordination period occurring on the incoming line $i \in L_u$ calculated as the sum of the values applied in the alternating headway,

N the number of connections with alternating headway applied on the incoming line $i \in L_u$ in the partial coordination period.

Then we calculate the latest time position of the last connection of the arriving line serving the transfer node $u \in U$ in the coordination period, from the relationship (6.11):

$$t_{ui1} + (n_{ui} - 1) \cdot \overline{T_i} + \overline{a_i} \tag{6.11}$$

where:

- t_{ui1} the earliest possible service time of the transfer node $u \in U$ by the first connection of the incoming line $i \in L_u$ in the coordination period
- n_{ui} number of the incoming line connections $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period
- $\overline{T_i}$ average value of the headway in the partial coordination period which is between the connections of incoming line applied $i \in L_u$
- $\overline{a_i}$ average value of the maximum permissible time shift on the coordinated line $i \in L$ when an alternating headway is applied on it

For the latest time position of the last connection of the departing line $j \in L_u$ also in the case of the occurrence of an alternating headway between the connections of the outgoing line, the following relationship (6.12) must be valid:

$$\begin{gathered} t_{uj1} + \left[\left(n_{uj} - 1\right) - 1\right] \cdot T_j + a_j < \\ < t_{ui1} + (n_{ui} - 1) \cdot \overline{T_i} + \overline{a_i} + tprest_{uij} \leq \\ \leq t_{uj1} + \left(n_{uj} - 1\right) \cdot T_j + a_j \end{gathered} \tag{6.12}$$

where:

- t_{ui1} the earliest possible service time of the transfer node $u \in U$ by the first connection of the incoming line $i \in L_u$ in the coordination period
- t_{uj1} the earliest possible service time of the transfer node $u \in U$ by the first connection of the departing line $j \in L_u$ in the coordination period

n_{ui}	number of the incoming line connections $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period
n_{uj}	number of the departing line connections $i \in L_u$ serving the transfer node $u \in U$ and included in the coordination period
$\overline{T_\iota}$	average value of the headway in the partial coordination period which is applied between the connections of incoming line $i \in L_u$
T_j	the value of the constant headway applied between the connections of the departing line $i \in L_u$
$\overline{a_\iota}$	average value of the maximum permissible time shift on the coordinated line when an alternating headway is applied on incoming line $i \in L_u$
a_j	value of the maximum permissible time shift of connections on a departing line $i \in L$
$tprest_{uij}$	the value of the passenger transfer time between the stations of the incoming line $i \in L_u$ and the point of departure of the departing line $j \in L_u$ at the transfer node $u \in U$

The preceding relationship can also be written in the form (6.13) after modification:

$$t_{uj1} + (n_{uj} - 2) \cdot T_j + a_j < t_{ui1} + (n_{ui} - 1) \cdot \overline{T_\iota} + \overline{a_\iota} + tprest_{uij} \leq \\ \leq t_{uj1} + (n_{uj} - 1) \cdot T_j + a_j \tag{6.13}$$

The solution of the inequality is analogous in the case of the occurrence of constant headways between the incoming and outgoing lines.

Example 6.8

At the transfer node $u \in U$ served by the set of lines L_u there is a time coordination of the connections of two lines $i \in L_u$ and $j \in L_u$, where $j \neq i$. First, the possible time position of the first connection of the line $i \in L_u$ is at time 4, the earliest possible time position of the first connection of the line $j \in L_u$ is at time 3, i.e $t_{ui1} = 4$ and $t_{uj1} = 3$. Connections on the incoming line $i \in L_u$ are run in alternating 7 and 8 minute headways, which are repeated regularly (in any order), connections on the outgoing *line $j \in L_u$ are run in a constant 6 minute headway, i.e $\overline{T_\iota} = 7{,}5$ minutes and $T_j = 6$ minutes. We assume the value of the transfer time between the stations at which the connections of coordinated lines stop to be $tprest_{uij} = 5$ minutes and the elementary time units in terms of time coordination are equal to 1 minute for both lines. Furthermore, we assume that we use all permissible time positions of connections within the headways (permissible time positions are integer values of time shifts of connections) for possible time shifts of connections in the case of both lines. The task is to calculate the minimum number of connections on the lines included in the coordination at the transfer node $u \in U$.*

We start the solution of the example by calculating the values of the maximum allowed time delays of the line connections $i \in L_u$ and $j \in L_u$. Because on the incoming line $i \in L_u$ an alternating headway is applied, it is necessary to calculate the average value of the maximum allowed time delay. Since in the case of both lines we use all the allowed time positions of the connections within the headways, then in the case of the incoming line $i \in L_u$ it is $\overline{a_\iota} = 6{,}5$ minutes and in the case of line $j \in L_u$ it is $a_j = 5$ minutes. With regard to the elementary time unit in terms of time coordination of 1 minute (thus only positions of connections in whole minutes are allowed), the average value of the maximum allowed time shift must be $6{,}5$ minutes calculated for line $i \in L_u$ and should be understood as a theoretical value which will never be applied in reality and which is only used to calculate the minimum number of connections on line $j \in L_u$.

The solution continues by calculating the value of the coordination period. The smallest common multiple of the numbers 7.5 and 6 is the number 30, so the value of the coordination period will be $K_p = 30$ minutes.

In the next procedure we calculate the number of connections of the incoming line $i \in L_u$ included in the coordination at the transfer node $u \in U$, see the relationship (6.10).

$$n_{ui} = \frac{N \cdot K_p}{T_i^p} = \frac{2 \cdot 30}{15} = 4$$

Then we calculate the latest time position of the last connection of the arriving line in the coordination period from the relationship (6.11)

$$t_{ui1} + (n_{ui} - 1) \cdot \overline{T_i} + \overline{a_i} = 4 + (4 - 1) \cdot 7{,}5 + 6{,}5 = 33$$

and the time position when passengers from the last connection of the arriving line serving the transfer node at the latest time position in the coordination period reach the stop of the connection *of the departing line from the relationship:*

$$t_{ui1} + (n_{ui} - 1) \cdot T_i + a_i + tprest_{uij} = 33 + 5 = 38$$

We then calculate the time position of the latest connection of the outgoing line from the relationship (6.13):

$$t_{uj1} + \left(n_{uj} - 2\right) \cdot T_j + a_j < t_{ui1} + (n_{ui} - 1) \cdot \overline{T_i} + \overline{a_i} + tprest_{uij} \leq$$
$$\leq t_{uj1} + \left(n_{uj} - 1\right) \cdot T_j + a_j$$

So we are looking for the value of n_{uj}*, for which*

$$3 + \left(n_{uj} - 2\right) \cdot 6 + 5 < 38 \leq 3 + \left(n_{uj} - 1\right) \cdot 6 + 5$$

The intersection of the intervals that are solutions of both inequalities is the interval

$$6 \leq n_{uj} < 7$$

and the minimum number of connections of the outgoing line corresponds to an integer value lying in the given interval. Thus, the minimum number of connections of an outgoing line included in the time coordination within the coordination period is $n_{uj} = 6$*. The coordination period therefore contains 6 connections of the outgoing line.*

In the next example, the alternating headway will be applied between the connections of the outgoing line.

Example 6.9

At the transfer node $u \in U$ *served by the set of lines* L_u *there is a time coordination of the connections of two lines* $i \in L_u$ *and* $j \in L_u$*, where* $i \neq j$*. First, the possible time* *position of the first connection of the incoming line* $i \in L_u$ *is at time 1, the earliest possible time position of the first connection of the outgoing line* $j \in L_u$ *is at time 7, i.e* $t_{ui1} = 1$ *and* $t_{uj1} = 7$*. Connections on the incoming line* $i \in L_u$ *are run on a constant 9-minute headway, connections on the outgoing line* $j \in L_u$ *are run in alternating 12 and 13 minute headways, which are repeated regularly (in any order), i.e* $T_i = 9$ *minutes and* $\overline{T_j} = 12{,}5$ *minutes. We assume the value of the transfer time between the stations at which the coordinated lines stop to be* $tprest_{uij} = 5$ *minutes and the elementary time units in terms of time coordination are equal to 1 minute for both lines. Furthermore, we assume that we use all permissible time positions of connections within the headways (permissible time positions are integer values of time shifts of connections) for possible time shifts of connections in the case of both lines. The task is to calculate the minimum number of connections on the lines included in the coordination at the transfer node* $u \in U$*.*

We start the solution of the example by calculating the values of the maximum allowed time delays of the line connections $i \in L_u$ *and* $j \in L_u$*. Because on the outbound line* $j \in L_u$ *an alternating headway is applied, it is necessary to calculate the average value of the maximum allowed time delay. Since in the case of both lines we use all permissible time positions of the connections within the headway values, then in the case of line* $i \in L_u$ *it is* $a_i = 8$ *minutes and in the case of the line* $j \in L_u$ *it is* $\overline{a_j} = 11{,}5$ *minutes. The average value of the maximum allowed time shift (delay) of* $11{,}5$ *minutes calculated for line* $j \in L_u$ *is again to be understood as a theoretical value which will never be applied in reality and which serves only to calculate the minimum number of connections on line* $j \in L_u$*.*

The solution continues by calculating the value of the coordination period. The smallest common multiple of the numbers 9 and 12.5 is the number 225, so the value of the coordination period will be $K_p = 225$ *minutes.*

In the next procedure we calculate the number of connections of the incoming line $i \in L_u$ *included in the coordination at the transfer node* $u \in U$*, see the relationship (6.10):*

$$n_{ui} = \frac{K_p}{T_i} = \frac{225}{9} = 25$$

Then we calculate the latest time position of the last connection of the arriving line in the coordination period from the relationship (6.11):

$$t_{ui1} + (n_i - 1) \cdot T_i + a_i = 1 + (25 - 1) \cdot 9 + 8 = 225$$

and the time position when passengers from the last connection of the arriving line serving the transfer node at the latest time position in the coordination period reach the stop of the connection of the departing line, from the relationship:

$$t_{ui1} + (n_i - 1) \cdot T_i + a_i + tprest_{uij} = 225 + 5 = 230$$

We then calculate the time position of the latest connection of the outgoing line from the relationship (6.13):

$$t_{uj1} + (n_j - 2) \cdot \overline{T_j} + \overline{a_j} < t_{ui1} + (n_i - 1) \cdot T_i + a_i + tprest_{uij} \\ \leq t_{uj1} + (n_j - 1) \cdot \overline{T_j} + \overline{a_j}$$

So we are looking for the value of n_j, for which it is valid

$$7 + (n_j - 2) \cdot 12{,}5 + 11{,}5 < 230 \leq 7 + (n_j - 1) \cdot 12{,}5 + 11{,}5$$

The intersection of the intervals that are solutions of both inequalities is the interval

$$17{,}92 \leq n_{uj} < 18{,}92$$

and the minimum number of connections of the outgoing line corresponds to an integer value lying in the given interval. Thus, the minimum number of connections of an outgoing line included in the time coordination within the coordination period is $n_{uj} = 18$. The coordination period therefore contains 18 connections of the outgoing line.

6.5 Methodology for creating optimization models of network node time coordination with applied headway

Before designing a mathematical model of network node time coordination with the application of headway, it is necessary to:

1. determination of the length of the coordination period, i.e. the time period during which time coordination will be performed at the transfer nodes,

2. determination of the minimum number of connections included in the coordination period,

3. ensuring time continuity of connections on the routes of lines at transfer nodes (i.e. ensuring that no unwanted time delays occur for connections serving individual lines at transfer nodes),

4. ensuring that coordination takes place within the same timeframe of a given coordination period.

Problem point 1 - determination of the length of the coordination period was addressed in subsection 6.4.1,

Problem point 2 - determining the minimum number of connections was addressed in subsection 6.4.2,

Problem point 3 - ensuring the continuity of time of connections on the routes of lines at transfer nodes is implemented by taking into account the travel times of connections between transfer nodes.

Problem point 4 - ensuring that the coordination takes place in the same time period will be implemented by projecting the time positions of the connections of the coordinated lines into the interval $\langle 0; K_p \rangle$ at all transfer nodes included in the network node time coordination. Points 3 and 4 will be addressed simultaneously.

The basic task is to set the earliest possible times for handling the transfer nodes.

First, a demonstration example will be given to show how to set the earliest possible service times of the transfer nodes served in a prescribed order corresponding to the direction of the connections of a coordinated line with a constant headway between the connections. A demonstration example will then be given to show how to set the service times of the transfer nodes served in a prescribed order corresponding to the direction of the connections of a coordinated line with an alternating headway between the connections.

<u>Setting the earliest possible service times of the transfer nodes of a line's connection with a constant headway between the connections.</u>

<u>*Example 6.10*</u>

Let us consider two transfer nodes with implemented coordination, between which the route of a line included in the coordination with a constant headway value of 12 minutes is routed. Let us consider a travel time of 4 minutes for a coordinated line between the two transfer nodes. The task is to identify the earliest possible service times of the transfer nodes by the first connection of a given line in the coordination period.

We start the solution by setting the first possible service time of the transfer node served as the second in the sequence to 0. If the earliest possible service time of the transfer node served as the second in the sequence is set in this way and the value of the constant headway of 12 minutes between connections, the service times of the transfer node by the next connections occur in times of 12, 24, etc. (the value of the service time by the last connection results from the value of the length of the coordination period). In order to calculate the earliest possible service times of the transfer node served first in the sequence, we must subtract the travel time values between the two transfer nodes from the earliest possible service times of the transfer node served second in the sequence, which is clearly shown in Table 6.1.

First, the possible service times of the transfer node served first in the sequence	*-4*	*8*	*20*	...
First, the possible service times of the transfer node served as the second in the sequence	*0*	*12*	*24*	...

Table 6.1: Calculation of the earliest possible times of service at the transfer node - two transfer nodes

From the values shown in Table 6.1 it can be seen that the connection serving the transfer node in a given direction as the second in sequence in time position 0 serves the transfer node served as the first in sequence in time position -4, which is the value before the start of the coordination period. In network node time coordination, however, it is not desirable to leave the coordination period, therefore the first connection serving the transfer node served as the first in the sequence should be considered the first connection lying inside the coordination period, i.e. the connection serving the transfer node in the earliest possible time position 8. While the connection serving the transfer node as the first in sequence at the earliest possible time position 8 can be considered as the first connection serving the transfer node, it is clear that the same connection will act as the second connection at the transfer node served as the second in sequence. For network node time coordination, however, the numbering of the sequence of connections serving specific transfer nodes is not relevant from the point of view of the calculation of the total time loss, but the time positions of the connections of specific transfer nodes by individual connections are relevant from the point of view of the calculation of the total time loss.

In the next example, we consider three transfer nodes.

<u>*Example 6.11*</u>

Let us consider three transfer nodes with implemented coordination, between which, by analogy with the previous example, the route of the line included in the coordination with a constant headway value of 12 minutes is routed. The transfer nodes mentioned *in the previous example will be considered as the transfer nodes served as the second in the sequence and the third in the sequence, between which the travel time on the route of the line is 4 minutes, the transfer node with coordination served as the first in the sequence will be considered as the transfer node from which the travel time to the transfer node served as the second in the sequence will be 10 minutes. The task is to identify the earliest possible service times of the transfer nodes by the first connection of a given line in the coordination period.*

The solution procedure will be analogous to the previous example.

We start the solution by setting the earliest possible service time of the transfer node, which is served as the third in the sequence, by the first service to 0. If the earliest possible service time of the transfer node, which is served as the third in the sequence, is set in this way and the value of the constant headway between the connections, the service times of that transfer node by the next connections occur at times 12, 24, etc.. The first possible service times of the transfer node, which will be served as the second in the sequence, are taken from the previous Example 6.10, and first, the possible times of the transfer node, which will be served as the first in the sequence, are calculated analogously as in Example 6.10, i.e. we get the values which can be seen in Table 6.2.

The first possible service times of the transfer node served first in the sequence	*-14*	*-2*	*10*	…
The first possible service times of the transfer node served as the second in the sequence	*-4*	*8*	*20*	…
The first possible service times of the transfer node served as the third in the sequence	*0*	*12*	*24*	…

Table 6.2: Calculation of the earliest possible times for transfer node services - three transfer nodes

From the values shown in Table 6.2 it can be seen that the first connection serving the transfer node in a given direction as the third in sequence in time position 0 serves the transfer node served as the second in sequence in time position -4, which is the value before the start of the coordination period. The analogy is the same for the node served *first in the sequence, where the earliest possible service time is -14. As already mentioned in the commentary to the solved example 6.10, it is not desirable to leave the coordination period in network node time coordination. Therefore, the first connection serving the transfer node as the second in the sequence must be considered as the first connection lying inside the coordination period, i.e. the connection serving the transfer node in the earliest possible time position 8 and in the case of the transfer node served as the first in the sequence, the first connection lying inside the coordination period is the connection serving the transfer node in the earliest possible time position 10. From these earliest possible positions, the earliest possible time positions of the other connections in the coordination period are then derived.*

The procedure of setting the earliest possible service times of the transfer nodes of a line's connection with applied alternating headway between the connections will be demonstrated on the case of a two-value alternating headway.

<u>Setting the earliest possible service times of the transfer nodes of a line connection with an applied two-value alternating headway between the connections</u>

A two-value alternating headway means an operating situation where an alternating sequence of two headway values differing by α minutes. If there are more than two values of alternating headway rates, the analogy would be followed. In the case of an alternating headway, the procedure is to determine the so-called base headway value, which corresponds to the lowest value of the listed headways, to which values expressing quantifying deviations from the base headway value are added as necessary.

However, when alternating headway occur, it is necessary to ensure that the individual headway values are repeated periodically. This can be achieved in mathematical programming by using binary variables. In the following text we assume that the alternating headway consists of two periodically repeating values that differ by a value of $\alpha = 1$ minutes. In the model for each line $i \in L$ with the occurrence of an alternating headway and each of its directions $l \in S$ we introduce two auxiliary binary variables, namely v_{il} and w_{il}. These binary variables will allow the base headway values to be increased on the line $i \in L$ in the direction of $l \in S$. Thus, for example, if an alternating headway rate of 7, 8, 7, 8 minutes, etc. is applied on the line, the base headway has the value of $min\{7,8\} = 7$ minutes. The values of the binary variables for a given line and direction are then appropriately added to the value of the base headway to ensure that the desired alternating headway values are achieved.

<u>*Example 6.12*</u>

Procedure for working with variables v_{il} and w_{il} for line connections $i \in L$ running in the direction of $l \in S$ we will explain using the case of a transfer node in which the $t_{uil1} = 0$ is valid (i.e., it is a transfer node that is served last in the line route). In order to achieve an alternating headway, for the value of t_{uil2} , $t_{uil2} = t_{uil1} + +T_i + v_{il}$ will be valid. When the binary variable v_{il} takes on the value after the optimization calculation is completed $v_{il} = 0$, then the headway between the first and second connection at the last transfer node will be equal to the value of the base headway T_i. When the binary variable v_{il} takes on a value after the optimization calculation is completed $v_{il} = 1$, then the headway between the first and second connection at the last transfer node will be equal to the value of the base headway T_i increased by 1 (the increased base headway value will be used between the first and second connection). Variable value v_{il} is therefore intended to achieve an increased value of the base headway between the first and second connection. The variable w_{il} is also applied analogously, which is introduced into the model to achieve an increased value of the base headway between the second and third connection.

The model must then treat only one of these variables as taking the value 1, which is ensured by a group of constraints of the type $v_{il} + w_{il} = 1$ for each line with alternating headway and the corresponding direction.

The headways between successive pairs of connections must be set using binary variables so that the required headway values are regularly alternated. So in our example, it would be $t_{uil3} = t_{uil1} + 2 \cdot T_i + v_{il} + w_{il}$, $t_{uil4} = t_{uil1} + 3 \cdot T_i + 2 \cdot v_{il} + w_{il}$, $t_{uil5} = t_{uil1} + 4 \cdot T_i + 2 \cdot v_{il} + 2 \cdot w_{il}$ *etc.*

The earliest possible service times of the previous transfer nodes by the same line, analogous to the cases of constant headways rates on coordinated lines, must be set in such a way as to maintain the time continuity of the connections serving all transfer nodes on a given line with the occurrence of alternating headway rates in a given direction. The procedure is demonstrated in example 6.13.

<u>Example 6.13</u>

Consider a line with alternating 7 and 8 minute headways, with two transfer nodes A and B on its route. The value of the base headway is therefore T_i *=7 minutes. Consider a direction in which transfer node A is served first and then transfer node B. Let the travel time from node A to node B be 13 minutes and the coordination period be 30 minutes. Node B is served as the second (last) in the sequence, so we set the possible service times for it first. The service times will correspond to the values* t_{uil1}, t_{uil2}, t_{uil3}, t_{uil4} *and* t_{uil5} *given in the previous paragraph. So these will be the values* $t_{uil1} = 0$, $t_{uil2} = 7 + v_{il}$, $t_{uil3} = 14 + v_{il} + w_{il}$, $t_{uil4} = 21 + 2 \cdot v_{il} + w_{il}$ *and* $t_{uil5} = 28 + 2 \cdot v_{il} + 2 \cdot w_{il}$. *In the next procedure, we need to project these times to the transfer node A. Taking into account the travel time from transfer node A to transfer node B, which is 13 minutes, we find that in a coordination period of 30 minutes, the first connection serving transfer node A is the one serving transfer node B at* $14 + v_{il} + w_{il}$. *In order for the same connection to serve node B at* $14 + v_{il} + w_{il}$, *it must serve node A at*$(14 - 13) + v_{il} + w_{il} = 1 + v_{il} + w_{il}$. *This value must be used as a basis for determining the next values of the earliest possible service times of the transfer node A of a connection of line* $i \in L$. *For the second connection serving node A, the earliest possible service time will be set to* $8 + 2 \cdot v_{il} + w_{il}$, *for the third connection at the value*

$15 + 2 \cdot v_{il} + 2 \cdot w_{il}$, *for the fourth connection at the value* $22 + 3 \cdot v_{il} + 2 \cdot w_{il}$ *and finally for the fifth service* $29 + 3 \cdot v_{il} + 3 \cdot w_{il}$.

In the models presented below, only lines for which more than one transfer node with the required time coordination of connections occurring on their routes are included. As already mentioned, it does not make sense to include lines with one transfer node on their route in the coordination, as their timetables can be coordinated individually, but depending on the results of the previous network node timing coordination.

6.6 Creation of an optimization model of network node time coordination for the operational variant with the same constant values of headways between connections on the arrival and departure of lines to/from all transfer nodes

This operational variant is the simplest operational variant. The problem formulation will take the following form:

A set of transfer nodes U and a set of lines L are determined, whose connections are to be coordinated in the transfer nodes in a defined way. For each node $u \in U$ a set of lines is further defined L_u whose connections $u \in U$ are to be coordinated in the node. For each line $i \in L$ a set of directions S_i is defined in which its connections are routed (in the following text we assume that the coordinated lines are of a shuttle character, i.e. the sets of directions are the same for all lines - there are two, so it is possible to drop the index in the case of sets S_i and thus in the following text we will work only with a simplified notation S). For each line $i \in L$ and its direction $l \in S$ a set of connections P_{il} is defined to be coordinated. At each transfer node $u \in U$ included in the coordination, there will be 1 connection for each arriving line $i \in L_u$ and direction $l \in S$ included in the model, i.e., $P_{il} = \{1\}$ and for each outgoing line $j \in L_u$ and direction $s \in S$ 2 connections will be included in the model, i.e, $P_{js} = \{1; 2\}$.

Each coordination requirement in the addressed network is defined by an ordered seven $[u; i; l; k, j; s; f]$, where $u \in U$, $i \in L_u$ and $j \in L_u$, $l \in S$, $s \in S$ and $k \in P_{il}$. The first number in the ordered seven identifies the coordinating node, the second number represents the number of the incoming line whose connections are to be coordinated, the third number represents the direction number of the incoming line whose connections are to be coordinated, and the fourth number represents the number of the connection of the incoming line from which a transfer is requested. The fifth and sixth numbers represent the number of the departing line and its direction to which the transfer is requested. It is logical that it must be true that $j \neq i$, because coordinating transfers between connections of the same line is not relevant in practice. However, concerning the directions of line connections, it may also be the case that $l = s$ as it is true that a transfer between different lines may be required in the same direction. The last number in the seven represents the volume of passengers changing at the node $u \in U$ from the connection $k \in P_{il}$ of line $i \in L_u$ going in the direction of $l \in S$ to line connection $j \in L_u$ going in the direction of $s \in S$.

The existence of the coordination requirement in the mathematical model is defined by the matrix $\boldsymbol{B}$ containing the values 0 or 1. When at a node $u \in U$ there is a request to create a coordination from the incoming line connection $i \in L_u$ travelling in the direction of $l \in S$ to the connections of line $j \in L_u$ going in the direction of $s \in S$, then $b_{uiljs} = 1$, otherwise $b_{uiljs} = 0$.

For each pair of lines $i \in L_u$ and $j \in L_u$where $u \in U$, in the situation where $b_{uiljs} = 1$, the value of the transfer time is defined as $tprest_{uij}$ (it is assumed that the value of the transfer time between the connections of the coordinated lines does not depend on which pair of connections of the given lines is involved) and the passenger volume f_{uilkjs} transferring (for the selected coordination period) from a connection $k \in P_{il}$ of line $i \in L_u$ going in the direction of $l \in S$ to a line connection $j \in L_u$ going in the direction of $s \in S$. The volume of transferring passengers is defined in this case as the average number of passengers transferring from all connections of the line $i \in L_u$ during the coordination period.

For each incoming line $i \in L_u$, of which the connections serving the coordination node $u \in U$ in the direction of $l \in S$ is being transferred, the earliest possible time the node can be served by the first (and only) connection of that line is defined as t_{uil1}, for each departing line $j \in L_u$ whose connections serving the coordination node $u \in U$ in the direction of $s \in S$ is being transferred, a headway is defined as T_j (the regular time interval between two connections of the same line serving a route in the same direction) and the earliest possible time for the transfer node to be served by two connections of the line t_{ujs1} and t_{ujs2}, whereby $t_{ujs2} = t_{ujs1} + T_j$ is valid.

The task is to decide on the time shifts of line connections in individual directions coordinated at the respective nodes so that the time shifts of connections of individual lines running in the same direction are uniform (to maintain the values of the prescribed headways on the lines) and at the same time to minimize the total time loss of all transferring passengers.

In order to model the decision, we introduce the following variables into the optimization task:

x_{il} non-negative variable modelling the time shift of all connections of the line $i \in L$ in the direction $l \in S$ calculated from their earliest possible time positions,

h_{uilkjs} a non-negative variable modelling the time loss of each passenger transferring at the transfer node $u \in U$ from a connection $k \in P_{il}$ of line $i \in L_u$ travelling in the direction of $l \in S$ to the nearest line connection $j \in L_u$ going in the direction of $s \in S$,

$z_{uilkjsp}$ auxiliary binary variable modelling the formation of the coordination link between the connection $k \in P_{il}$ of the incoming line $i \in L_u$ travelling in the direction of $l \in S$ and the connection $p \in P_{js}$ of the outgoing line $j \in L_u$ going in the direction of $s \in S$ at the transfer node $u \in U$.

The symbol M represents the enough high constant. For its value it is possible to select for example, a number 10^6.

The mathematical model of the solved problem will have the form (6.14) - (6.21):

$$min\, f(x,h,z) = \sum_{u \in U} \sum_{i \in L_u} \sum_{l \in S} \sum_{k \in P_{il}} \sum_{\substack{j \in L_u \\ j \neq i}} \sum_{s \in S} b_{uiljs} \cdot f_{uilkjs} \cdot h_{uilkjs} \tag{6.14}$$

subject to:

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$$[t_{ujsp} + x_{js}] - [t_{uilk} + x_{il} + tprest_{uij}] \geq \geq M \cdot (z_{uilkjsp} - 1) \quad \text{for } u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in \{1\}, p \in \{1; 2\} \text{ and } b_{uiljs} = 1 \tag{6.15}$$

$$[t_{ujsp} + x_{js}] - [t_{uilk} + x_{il} + tprest_{uij}] \leq \leq h_{uilkjs} + M \cdot (1 - z_{uilkjsp}) \quad \text{for } u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in \{1\}, p \in \{1; 2\} \text{ and } b_{uiljs} = 1 \tag{6.16}$$

$$\sum_{p \in \{1;2\}} z_{uilkjsp} = 1 \quad \text{for } u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in \{1\} \text{ and } b_{uiljs} = 1 \tag{6.17}$$

$$x_{il} \leq a_{il} \quad \text{for } i \in L \text{ and } l \in S \tag{6.18}$$

$$x_{il} \in R_0^+ \quad \text{for } i \in L \text{ and } l \in S \tag{6.19}$$

$$h_{uilkjs} \in R_0^+ \quad \text{for } u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in \{1\} \text{ and } b_{uiljs} = 1 \tag{6.20}$$

$$z_{uilkjsp} \in \{0; 1\} \quad \text{for } u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in \{1\}, p \in \{1; 2\} \text{ and } b_{uiljs} = 1 \tag{6.21}$$

The function (6.14) represents the optimization criterion - the total time loss of all transferring passengers at all transfer nodes. The group of constraints (6.15) ensures that in the case of temporal inadmissibility of the positions of connections of coordinated lines running in the directions affected by the coordination, no coordination link is created. The group of constraints (6.16) quantifies the time loss of transferring passengers generated by the emergence of coordination links. The group of constraints (6.17) will ensure the formation of coordination links. The group of constraints (6.18) shall ensure that any connection time shifts generated to reduce the overall time loss do not exceed the maximum allowable time shift values. The groups of constraints (6.19) - (6.21) define the domains of definition of the variables used in the model.

6.7 Methodology for creating an optimization model for the operational variant with different values of constant headways between connections on the arrival and departure of lines to/from transfer nodes

The problem formulation for the operational variant solved in subsection 6.7 will be as follows:

A set of transfer nodes is given U and the set of lines L whose connections are to be coordinated in the transfer nodes in a defined way. For each node $u \in U$ a set of lines is further defined L_u whose connections $u \in U$ are to be coordinated in the node. For each line $i \in L$ a set of directions S_i is defined in which its connections are routed (in the following text we assume again that the coordinated lines are of a shuttle character, i.e. the sets of directions are the same for all lines - there are two, so it is possible to drop the index in the case of sets S_i and thus in the following text we will work only with a simplified notation S). For each line $i \in L$ and direction $l \in S$ a set of connections P_{il} is defined that will be coordinated. In contrast to the previous case, the number of connections included in the coordination is the number of connections in the coordination period corresponding to the smallest common multiple of all node coordination periods, the details for calculating the coordination period are given in subsection 6.4.

Each coordination requirement in the addressed network is defined by an ordered seven $[u; i; l; k, j; s; f]$, where $u \in U$, $i \in L_u$ and $j \in L_u$, $l \in S$, $s \in S$ and $k \in P_{il}$. The first number in the ordered seven identifies the coordinating node, the second number represents the number of the incoming line whose connections are to be coordinated, the third number represents the direction number of the incoming line whose connections are to be coordinated, and the fourth number represents the number of the connection of the incoming line from which a transfer is requested. The fifth and sixth numbers represent the number of the departing line and its direction to which the transfer is requested. It is logical that, again, it must be true that $j \neq i$, because coordinating transfers between connections of the same lines is not relevant in practice. However, in the case of the directions of line connections, it may also be the case that $l = s$ is valid, as it is true that a transfer between different lines may be required in the same direction. The last number in the seven represents the number of passengers changing at the transfer node $u \in U$ from the connection $k \in P_{il}$ of line $i \in L_u$ going in the direction of $l \in S$ to a line connection $j \in L_u$ going in the direction of $s \in S$.

The existence of the coordination requirement in the mathematical model is again defined by the matrix $\boldsymbol{B}$ containing the values 0 or 1. When at a transfer node $u \in U$ there is a request to create a coordination link from a connection of the incoming line $i \in L_u$ travelling in the direction of $l \in S$ to the connection of line $j \in L_u$ going in the direction of $s \in S$, then $b_{uiljs} = 1$, otherwise $b_{uiljs} = 0$.

For each pair of lines $i \in L_u$ and $j \in L_u$ where $u \in U$ in the situation where $b_{uiljs} = 1$, the value of the transfer time is defined as $tprest_{uij}$ (again, it is assumed that the value of the transfer time between connections of coordinated lines does not depend on which pair of lines is involved). However, in case of passenger volumes f_{uilkjs} transferring (for the selected coordination period) from the connection $k \in P_{il}$ of line $i \in L_u$ going in the direction of $l \in S$ to a line connection $j \in L_u$ going in the direction of $s \in S$ the procedure must be different. The reason for the different approach is the existence of a larger number of incoming connections serving transfer nodes. In such a case, it is necessary to proceed differently from the procedure described in subsection 6.6, usually in one of two ways.

The first method is to carry out a detailed transport survey to determine the passenger volumes arriving on each of the connections arriving at the transfer nodes, monitoring the directions to which passengers transfer in addition to the numbers of passengers arriving on the lines arriving at the transfer nodes. However, the pitfall of the first approach is that changes in volumes will occur when connections are moved to other time positions, because moving a connection to a different time position may make that connection temporally irrelevant to the passenger.

If the results of the directional traffic survey are not available, it is possible to proceed by working with the values of the transferring passenger volumes burdened with the maximum level of uncertainty, i.e. the number of passengers transferring between the connections of two lines during the coordination period will be divided evenly between the arriving connections (i.e. the total volume of transferring passengers at a given transfer node in a given transfer link will be divided by the number of connections of the arriving line included in the coordination within the coordination period). The formation of an integer value of the volume of transferring passengers is not binding, so it is possible to work with non-integer values of volumes (these are average values).

For each incoming line $i \in L_u$, of which the connections serving the coordination node $u \in U$ in the direction of $l \in S$ are being transferred, the earliest possible time the node can be served by the first connection of that line is defined as t_{uil1}, the value of the headway T_i and then the earliest possible arrival times of the connections in the coordination period, which can be calculated from the relationship $t_{uilk} = t_{uil1} + (k-1) \cdot T_j$, where $k \in P_{il} \setminus \{1\}$. For each departing line $j \in L_u$, whose connections serving the coordinating node $u \in U$ in the direction of $s \in S$ are being transferred, the value of the headway is T_j, the earliest possible time for the transfer node to be served by the first departing connection of that line is t_{ujs1} and then the earliest possible departure times of the other connections in the coordination period can be calculated from the relationship $t_{ujsp} = t_{ujs1} + (p-1) \cdot T_j$, where $p \in P_{js} \setminus \{1\}$.

The task is to decide on the time shifts of line connections in individual directions coordinated at the respective nodes so that the time shifts of connections of individual lines running in the same direction are uniform (to maintain the values of the prescribed headway on the lines) and at the same time to minimize the total time loss of all transferring passengers.

In order to model the decision, we introduce the following variables into the optimization task:

x_{il} non-negative variable modelling the time shift of all connections of the line $i \in L$ in the direction $l \in S$ calculated from their earliest possible time positions,

h_{uilkjs} a non-negative variable modelling the time loss of each passenger transferring at the transfer node $u \in U$ from a connection $k \in P_{il}$ of line $i \in L_u$ travelling in the direction of $l \in S$ to the nearest line connection $j \in L_u$ going in the direction of $s \in S$,

$z_{uilkjsp}$ auxiliary binary variable modelling the formation of the coordination link between the connection $k \in P_{il}$ of the incoming line $i \in L_u$ travelling in the direction of $l \in S$ and the connection $p \in P_{js}$ of the outgoing line $j \in L_u$ going in the direction of $s \in S$ at the transfer node $u \in U$.

The symbol M represents the enough high constant. For its value it is possible to select, for example, a number such as 10^6.

The mathematical model of the solved problem will have the form (6.22) - (6.29):

$$min\, f(x,h,z) = \sum_{u\in U} \sum_{i\in L_u} \sum_{l\in S} \sum_{k\in P_{il}} \sum_{\substack{j\in L_u \\ j\neq i}} \sum_{s\in S} b_{uiljs} \cdot f_{uilkjs} \cdot h_{uilkjs} \tag{6.22}$$

subject to:

$$\left[t_{ujsp} + x_{js}\right] - \left[t_{uilk} + x_{il} + tprest_{uij}\right] \geq \geq M \cdot \left(z_{uilkjsp} - 1\right) \quad \text{for } u \in U,\, i \in L_u,\, j \in L_u,\, j \neq i,\, l \in S,\, s \in S,\, k \in P_{il},\, p \in P_{js},\, b_{uiljs} = 1 \tag{6.23}$$

$$[t_{ujsp} + x_{js}] - [t_{uilk} + x_{il} + tprest_{uij}] \leq \leq h_{uilkjs} + M \cdot (1 - z_{uilkjsp})$$ for $u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in P_{il}, p \in P_{js}, b_{uiljs} = 1$ (6.24)

$$\sum_{p \in P_{js}} z_{uilkjsp} = 1$$ for $u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in P_{il}, b_{uiljs} = 1$ (6.25)

$$x_{il} \leq a_{il}$$ for $i \in L$ a $l \in S$ (6.26)

$$x_{il} \in R_0^+$$ for $i \in L$ a $l \in S$ (6.27)

$$h_{uilkjs} \in R_0^+$$ for $u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in P_{il}, b_{uiljs} = 1$ (6.28)

$$z_{uilkjsp} \in \{0; 1\}$$ for $u \in U, i \in L_u, j \in L_u, j \neq i, l \in S, s \in S, k \in P_{il}, p \in P_{js}, b_{uiljs} = 1$ (6.29)

Function (6.22) represents the optimization criterion - the total time loss of all transferring passengers at all transfer nodes. The group of constraints (6.23) ensures that in the case of temporal inadmissibility of the positions of connections of coordinated lines running in the directions affected by the coordination, no coordination link is created. The group of constraints (6.24) quantifies the time loss of transferring passengers generated by the emergence of coordination links. The group of constraints (6.25) ensures the formation of coordination links. The group of constraints (6.26) shall ensure that any connection time shifts generated to reduce the overall time loss do not exceed the maximum allowable time shift values. The groups of constraints (6.27) - (6.29) define the domains of definition of the variables used in the model.

6.8 Methodology for creating an optimization model for the operational variant with a constant value of the headway between the connections on the arrival to the transfer node and an alternating value of the headway between the connections on the departure from the transfer node

The problem formulation for the operational variant presented in subsection 6.8 will be as follows:

A set of transfer nodes U and the set of lines L are specified, whose connections are to be coordinated in the transfer nodes in a defined way. For each node $u \in U$ a set of lines is further defined L_u whose connections are to be coordinated in the node $u \in U$. For each line $i \in L$ a set of directions is defined as S_i in which its connections are routed (in the following text we assume again that the coordinated lines are of a shuttle character, i.e. the sets of directions are the same for all lines - there are two, so it is possible to drop the index in the case of sets S_i and thus in the following text we will only work with a simplified notation S). For each line $i \in L$ and direction $l \in S$ a set of connections P_{il} is defined to be coordinated. Analogously, as in the previous case, the number of connections that corresponds to the number of connections in the coordination period corresponding to the smallest common multiple of the values of all node coordination periods is included in the coordination, the details for calculating the coordination period are given in subsection 6.4.

Each coordination requirement in the addressed network is defined by an ordered seven $[u; i; l; k, j; s; f]$, where $u \in U$, $i \in L_u$ and $j \in L_u^*$, $l \in S$, $s \in S$ and $k \in P_{il}$. The first number in the ordered seven identifies the coordinating node, the second number represents the number of the incoming line whose connections are to be coordinated, the third number represents the direction number of the incoming line whose connections are to be coordinated, and the fourth number represents the number of the incoming connection from which a transfer is requested. The fifth and sixth numbers represent the number of the departing line and its direction to which the transfer is requested. It is logical that, again, it must be true that $j \neq i$ is valid, because coordinating transfers between connections of the same lines is not relevant in practice. However, in the case of the directions of line connections, it may also be the case that $l = s$ as it is true that a transfer between different lines may be required to the same direction. The last number in the seven represents the number of passengers changing at the node $u \in U$ from the connection $k \in P_{il}$ of line $i \in L_u$ going in the direction of $l \in S$ to line connection $j \in L_u^*$ going in the direction of $s \in S$.

The existence of the coordination requirement in the mathematical model is again defined by the matrix $\boldsymbol{B}$ containing the values 0 or 1. When at node $u \in U$ there is a request to create a coordination link from the incoming line connection $i \in L_u$ going in the direction of $l \in S$ to the connections of line $j \in L_u$ going in the direction of $s \in S$, then $b_{uiljs} = 1$, otherwise $b_{uiljs} = 0$.

For each pair of lines $i \in L_u$ and $j \in L_u^*$ where $u \in U$, in the situation where $b_{uiljs} = 1$, the value of the transfer time is defined as $tprest_{uij}$ (it is again assumed that the value of the transfer time between the connections of coordinated lines does not depend on which pair of connections of the given lines is involved). In case of passenger volumes f_{uilkjs} transferring (for the selected coordination period) from the connection $k \in P_{il}$ of line $i \in L_u$ going in the direction of $l \in S$ to a line connection $j \in L_u$ going in the direction of $s \in S$ however, the procedure must again be different from the case described in subsection 6.6. The reason for the different approach is again the existence of a larger number of incoming connections arriving at transfer nodes. As a rule, one of two approaches is followed.

The first approach is again to carry out a detailed transport survey, which will identify the passenger volumes arriving on each of the connections of the incoming lines to the transfer nodes, monitoring not only the numbers of passengers arriving on the connections of the incoming lines to the transfer nodes, but also the directions to which passengers transfer.

The second approach is again to determine the transferring passenger volumes under conditions of maximum uncertainty, i.e. the number of passengers transferring between the connections of two lines per coordination period will be distributed evenly between the arriving connections.

For each incoming line $i \in L_u$, of which the connections serving the coordination node $u \in U$ in the direction of $l \in S$ are being transferred, the earliest possible time the node can be served by the first connection of that line is defined as t_{uil1}, the value of the headway as T_i and then the earliest possible arrival times of the connections in the coordination period can be calculated from the relationship $t_{uilk} = t_{uil1} + (k-1) \cdot T_i$, where $k \in P_{il} \setminus \{1\}$. For each departing line $j \in L_u^*$, whose connections serving the coordinating node $u \in U$ in the direction of $s \in S$ are transferred, the basic headway values are defined as T_j and the earliest possible time for the transfer node to be served by the first outbound connection of that line is t_{ujs1} and the earliest possible departure times of the other connections in the coordination period can be calculated from the relationship using the procedure in subsection 6.5. To simplify the symbolic notation of the model, let's first denote the possible service times of node $u \in U$ by the connection $p \in P_{js}$ of line $j \in L_u$ going in the direction $s \in S$ with the symbol $\tau_{ujsp}(v, w)$.

The task is to decide on the time shifts of line connections in individual directions coordinated at the respective nodes so that the shifts of connections of individual lines running in the same direction are uniform (to maintain the values of the prescribed headways on the lines) and to minimize the total time loss (TTL) of all transferring passengers.

In order to model the decision, we introduce the following variables into the optimization task:

x_{il} non-negative variable modelling the time shift of all connections of the line $i \in L$ in the direction $l \in S$ calculated from their earliest possible time positions,

h_{uilkjs} a non-negative variable modelling the time loss of each passenger transferring at the transfer node $u \in U$ from a connection $k \in P_{il}$ of line $i \in L_u$ travelling in the direction of $l \in S$ to the nearest line connection $j \in L_u$ going in the direction of $s \in S$,

$z_{uilkjsp}$ auxiliary binary variable modelling the formation of the coordination link between the connection $k \in P_{il}$ of the incoming line $i \in L_u$ travelling in the direction of $l \in S$ and the connection $p \in P_{js}$ of the outgoing line $j \in L_u$ going in the direction of $s \in S$ at the transfer node $u \in U$,

v_{js} auxiliary binary variable introduced to increase the value of the base headway by 1 elementary time unit after each even connection of the departing line $j \in L_u^*$ in the direction $s \in S$ within the coordination period serving the last transfer node on the line route,

w_{js} auxiliary binary variable introduced to increase the value of the base headway by 1 elementary time unit after each odd connection of the departing line $j \in L_u^*$ in the direction $s \in S$ within the coordination period serving the last transfer node on the line route.

The symbol M represents the enough high constant. For its value it is possible to select, for example, a number such as 10^6.

The mathematical model of the task will be of the form (6.30) - (6.40):

$$min\, f(x,h,z,v,w) = \sum_{u \in U} \sum_{i \in L_u} \sum_{l \in S} \sum_{k \in P_{il}} \sum_{\substack{j \in L_u^* \\ j \neq i}} \sum_{s \in S} b_{uiljs} \cdot f_{uilkjs} \cdot h_{uilkjs} \qquad (6.30)$$

subject to:

$$\left[\tau_{ujsp}(v,w) + x_{js}\right] - \\ -\left[t_{uilk} + x_{il} + tprest_{uij}\right] \geq \\ \geq M \cdot \left(z_{uilkjsp} - 1\right) \qquad \text{for } u \in U,\, i \in L_u,\, j \in L_u^*,\, j \neq i,\, l \in S,\, s \in S,\, k \in P_{il},\, p \in P_{js} \text{ and } b_{uiljs} = 1 \qquad (6.31)$$

$$\left[\tau_{ujsp}(v,w) + x_{js}\right] - \\ -\left[t_{uilk} + x_{il} + tprest_{uij}\right] \leq \\ \leq h_{uilkjs} + M \cdot \left(1 - z_{uilkjsp}\right) \qquad \text{for } u \in U,\, i \in L_u,\, j \in L_u^*,\, j \neq i, l \in S,\, s \in S,\, k \in P_{il},\, p \in P_{js} \text{ and } b_{uiljs} = 1 \qquad (6.32)$$

$$\sum_{p \in P_{js}} z_{uilkjsp} = 1 \quad \text{for } u \in U, i \in L_u, j \in L_u^*, j \neq i, l \in S, s \in S, k \in P_{il} \text{ and } b_{uiljs} = 1 \tag{6.33}$$

$$v_{js} + w_{js} = 1 \quad \text{for } j \in L^* \text{ and } s \in S \tag{6.34}$$

$$x_{il} \leq a_{il} \quad \text{for } i \in L \text{ and } l \in S \tag{6.35}$$

$$x_{il} \in R_0^+ \quad \text{for } i \in L \text{ and } l \in S \tag{6.36}$$

$$h_{uilkjs} \in R_0^+ \quad \text{for } u \in U, i \in L_u, j \in L_u^*, j \neq i, l \in S, s \in S, k \in P_{il} \text{ and } b_{uiljs} = 1 \tag{6.37}$$

$$z_{uilkjsp} \in \{0; 1\} \quad \text{for } u \in U, i \in L_u, j \in L_u^*, j \neq i, l \in S, s \in S, k \in P_{il}, p \in P_{js} \text{ and } b_{uiljs} = 1 \tag{6.38}$$

$$v_{js} \in \{0; 1\} \quad \text{for } j \in L^* \text{ and } s \in S \tag{6.39}$$

$$w_{js} \in \{0; 1\} \quad \text{for } j \in L^* \text{ and } s \in S \tag{6.40}$$

The function (6.30) represents the optimization criterion - the total time loss of all transferring passengers at all transfer nodes. The group of constraints (6.31) ensures that in the case of temporal inadmissibility of the positions of connections of coordinated lines running in the directions affected by coordination, no coordination link is created. The group of constraints (6.32) quantifies the time loss of transferring passengers generated by the emergence of coordination links. The group of constraints (6.33) ensures the formation of coordination links. The group of constraints (6.34) is created only for connections from the set of outgoing lines L^* and their directions in which the alternating headway is applied between them (in the case of shuttle lines, it is assumed that when the alternating headway is applied in one direction, it is also applied in the opposite direction). The group of constraints (6.35) shall ensure that any connection time shifts generated to reduce the overall time loss do not exceed the maximum allowable time shift values. The groups of constraints (6.36) - (6.40) define the domains of definition of the variables used in the model.

6.9 Methodology of creating an optimization model for the operational variant with alternating value of the headway between the connections on the arrival to the transfer node and constant value of the headway between the connections on the departure from the transfer node

The problem formulation will be as follows for the operational variant presented in subsection 6.9:

A set of transfer nodes is given U and the set of lines L whose connections are to be coordinated in the transfer nodes in a defined way. For each node $u \in U$ a set of lines L_u is further defined whose connections are to be coordinated in the node $u \in U$. For each line $i \in L$ a set of directions is again defined as S_i, in which its connections are routed (in the following text we assume that the coordinated lines are of a shuttle character, so the sets of directions are the same for all lines - there are two and thus again it is possible to drop the index in the case of sets S_i and thus in the following text we will work only with a simplified notation S). For each line $i \in L$ and direction $l \in S$ a set of connections P_{il} is defined to be coordinated. Analogously, as in the previous case, the number of connections that corresponds to the number of connections in the coordination period corresponding to the smallest common multiple of the values of all node coordination periods is included in the coordination, the details for calculating the coordination period are given in subsection 6.4.

Each coordination requirement in the addressed network is defined by an ordered seven $[u; i; l; k, j; s; f]$, where $u \in U$, $i \in L_u^{**}$ and $j \in L_u$, $l \in S$, $s \in S$ and $k \in P_{il}$. The first number in the ordered seven identifies the coordinating node, the second number represents the number of the incoming line whose connections are to be coordinated, the third number represents the direction number of the incoming line whose connections are to be coordinated, and the fourth number represents the number of the connection of the incoming line from which a transfer is requested. The fifth and sixth numbers represent the number of the departing line and its direction to which the transfer is requested. It is logical that it must be true that $j \neq i$, because coordinating transfers between connections of the same line is not relevant in practice. However, in the case of the directions of line connections, it may also be the case that $l = s$ is valid as it is true that a transfer between different lines may be required in the same direction. The last number in the seven represents the number of passengers changing at the node $u \in U$ from the connection $k \in P_{il}$ of line $i \in L_u^{**}$ going in the direction of $l \in S$ to a line connection $j \in L_u$ going in the direction of $s \in S$.

The existence of the coordination requirement in the mathematical model is again defined by the matrix $\boldsymbol{B}$ containing the values 0 or 1. When at a node $u \in U$ there is a request to create a coordination link from the connections of an incoming line $i \in L_u^{**}$ going in the direction of $l \in S$ to the connections of a line $j \in L_u$ going in the direction of $s \in S$, then $b_{uiljs} = 1$, otherwise $b_{uiljs} = 0$.

For each pair of lines $i \in L_u^{**}$ and $j \in L_u$ where $u \in U$, in the situation where $b_{uiljs} = 1$, the value of the transfer time is defined as $tprest_{uij}$ (it is again assumed that the value of the transfer time between the connections of the coordinated lines does not depend on which pair of connections of the given lines is involved). In case of passenger volumes f_{uilkjs} transferring (for the selected coordination period) from the connection $k \in P_{il}$ of line $i \in L_u^{**}$ going in the direction of $l \in S$ to a line connection $j \in L_u$ going in the direction of $s \in S$ however, the procedure must be different again from the first case described in subsection 6.6. The reason for the different approach is again the existence of a larger number of incoming connections arriving at transfer nodes. As a rule, two procedures are used.

The first method is to carry out a detailed transport survey to determine the passenger volumes arriving on each of the connections arriving at the transfer nodes, monitoring the directions to which passengers transfer in addition to the numbers of passengers arriving on the lines arriving at the transfer nodes. However, the pitfall of the first approach is that changes in volumes will occur when connections are moved to other time positions, because moving a connection to a different time position may make that connection temporally irrelevant to the passenger.

If the results of the directional traffic survey are not available, it is possible to proceed by working with the values of transferring passenger volumes burdened with the maximum level of uncertainty, i.e. the number of passengers transferring between the connections of two lines per coordination period will be distributed evenly between the arriving connections.

For each incoming line $i \in L_u^{**}$, of which the connections serving the coordination node $u \in U$ in the direction of $l \in S$ are being transferred, the earliest possible time the node can be served by the first connection of that line is defined as t_{uil1}, the value of the basic headway is T_i and the earliest possible departure times of the other connections in the coordination period can be calculated from the relationship using the procedure in subsection 6.5. To simplify the symbolic notation of the model, let's first denote the possible times of the node $u \in U$ being served by the connection $k \in P_{il}$ of line $i \in L_u^{**}$ going in the direction $l \in S$ with the symbol $\tau_{uilk}(v, w)$. For each departing line $j \in L_u$ whose connections at coordinating node $u \in U$ in the direction of $s \in S$ are being transferred, the headway values are defined as T_j, the earliest possible service time for the transfer node to be served by the first departing connection of that line is t_{ujs1} and the earliest possible arrival times of the connections in the coordination period can be calculated from the relationship $t_{ujsp} = t_{ujs1} + (p-1) \cdot T_j$, where $k \in P_{js} \setminus \{1\}$.

The task is to decide on the time shifts of line connections in individual directions coordinated at the respective nodes so that the time shifts of connections of individual lines running in the same direction are uniform (to maintain the values of the prescribed headways on the lines) and at the same time to minimize the total time loss of all transferring passengers.

In order to model the decision, we introduce the following variables into the optimization task:

x_{il} non-negative variable modelling the time shift of all connections of the line $i \in L$ in the direction $l \in S$ calculated from their earliest possible time positions,

h_{uilkjs} a non-negative variable modelling the time loss of each passenger transferring at the transfer node $u \in U$ from a connection $k \in P_{il}$ of line $i \in L_u$ travelling in the direction of $l \in S$ to the nearest line connection $j \in L_u$ going in the direction of $s \in S$,

$z_{uilkjsp}$ auxiliary binary variable modelling the formation of the coordination link between the connection $k \in P_{il}$ of the incoming line $i \in L_u$ travelling in the direction of $l \in S$ and the connection $p \in P_{js}$ of the outgoing line $j \in L_u$ going in the direction of $s \in S$ at the transfer node $u \in U$,

v_{il} auxiliary binary variable introduced to increase the value of the base headway by α elementary time units after each even line connection $i \in L_u^{**} \cup L_u^{*}$ in the direction of $l \in S$ within the coordination period serving the last transfer node on the line's route,

w_{il} auxiliary binary variable introduced to increase the value of the base headway by α elementary time units after each odd line connection $i \in L_u^{**} \cup L_u^{*}$ in the direction of $l \in S$ within the coordination period, except for the first connection serving the last transfer node on the line's route.

The symbol M represents the enough high constant. For its value it is possible to select, for example, a number such as 10^6.

The mathematical model of the solved problem will have the form (6.41) - (6.51):

$$min\, f(x,h,z,v,w) = \sum_{u \in U} \sum_{i \in L_u^{**}} \sum_{l \in S} \sum_{k \in P_{il}} \sum_{\substack{j \in L_u \\ j \neq i}} \sum_{s \in S} b_{uiljs} \cdot f_{uilkjs} \cdot h_{uilkjs} \qquad (6.41)$$

subject to:

$$[t_{ujsp} + x_{js}] - [\tau_{uilk}(v,w) + x_{il} + \\ + tprest_{uij}] \geq M \cdot (z_{uilkjsp} - 1) \qquad \text{for } u \in U, i \in L_u^{**}, j \in L_u, j \neq i, l \in S, s \in S, k \in P_{il}, p \in P_{js} \text{ and } b_{uiljs} = 1 \qquad (6.42)$$

$$[t_{ujsp} + x_{js}] - [\tau_{uilk}(v,w) + x_{il} + \\ + tprest_{uij}] \leq h_{uilkjs} + M \cdot (1 - z_{uilkjsp}) \quad \text{for } u \in U,\, i \in L_u^{**},\, j \in L_u,\, j \neq i,\, l \in S,\, s \in S,\, k \in P_{il},\, p \in P_{js} \text{ and } b_{uiljs} = 1 \tag{6.43}$$

$$\sum_{p \in P_{js}} z_{uilkjsp} = 1 \quad \text{for } u \in U,\, i \in L_u^{**},\, j \in L_u,\, j \neq i,\, l \in S,\, s \in S,\, k \in P_{il} \text{ and } b_{uiljs} = 1 \tag{6.44}$$

$$v_{il} + w_{il} = 1 \quad i \in L^{**} \text{ a } l \in S \tag{6.45}$$

$$x_{il} \leq a_{il} \quad \text{for } i \in L \text{ and } l \in S \tag{6.46}$$

$$x_{il} \in R_0^+ \quad \text{for } i \in L \text{ and } l \in S \tag{6.47}$$

$$h_{uilkjs} \in R_0^+ \quad \text{for } u \in U,\, i \in L_u^{**},\, j \in L_u,\, j \neq i,\, l \in S,\, s \in S,\, k \in P_{il} \text{ and } b_{uiljs} = 1 \tag{6.48}$$

$$z_{uilkjsp} \in \{0; 1\} \quad \text{for } u \in U,\, i \in L_u^{**},\, j \in L_u,\, j \neq i,\, l \in S,\, s \in S,\, k \in P_{il},\, p \in P_{js} \text{ and } b_{uiljs} = 1 \tag{6.49}$$

$$v_{il} \in \{0; 1\} \quad \text{for } i \in L^{**} \text{ and } l \in S \tag{6.50}$$

$$w_{il} \in \{0; 1\} \quad \text{for } i \in L^{**} \text{ and } l \in S \tag{6.51}$$

The function (6.41) represents the optimization criterion - the total time loss of all transferring passengers at all transfer nodes. The group of constraints (6.42) ensures that in the case of temporal inadmissibility of the positions of connections of coordinated lines running in the directions affected by coordination, no coordination link is created. The group of constraints (6.43) quantifies the time loss of transferring passengers generated by the emergence of coordination links. The group of constraints (6.44) ensures the formation of coordination links. The group of constraints (6.45) is created only for connections in the set of incoming lines L^{**} and their directions in which the alternating headway is applied between them (in the case of shuttle lines, it is assumed that when the alternating headway is applied in one direction, it is also applied in the opposite direction). The group of constraints (6.46) shall ensure that any connection time shifts generated to reduce the overall time loss do not exceed the maximum allowable time shift values. The group of constraints (6.47) - (6.51) defines the domains of definition of the variables used in the model.

6.10 Methodology for creating an optimization model for the operational variant with alternating value of the headway between the connections on the arrival to the transfer node and alternating value of the headway between the connections on the departure from the transfer node

The problem formulation will be as follows for the operational variant presented in subsection 6.10:

A set of transfer nodes U and a set of lines L are specified whose connections are to be coordinated in the transfer nodes in a defined way. For each node $u \in U$ a set of lines is further defined as L_u whose connections are to be coordinated in the node $u \in U$. For each line $i \in L$ a set of directions S_i is defined, in which its connections are routed (in the following text we assume that these are shuttle lines, i.e. the sets of directions are the same for all lines - there are two, so it is possible to drop the index in the case of sets S_i and thus in the following text we will work only with a simplified designation S). For each line $i \in L$ and direction $l \in S$ a set of connections P_{il} to be coordinated is defined . Analogously, as in the previous case, the number of connections that corresponds to the number of connections in the coordination period corresponding to the smallest common multiple of the values of all node coordination periods is included in the coordination, the details for calculating the coordination period are given in subsection 6.4.

Each coordination requirement in the addressed network is defined by an ordered seven $[u; i; l; k, j; s; f]$, where $u \in U$, $i \in L_u^{**}$ and $j \in L_u^{*}$, $l \in S$, $s \in S$ and $k \in P_{il}$. The first number in the ordered seven identifies the coordinating node, the second number represents the number of the incoming line whose connections are to be coordinated, the third number represents the direction number of the incoming line whose connections are to be coordinated, and the fourth number represents the number of the connection of the incoming line from which a transfer is requested. The fifth and sixth numbers represent the number of the departing line and its direction to which the transfer is requested. It is logical that, again, it must be true that $j \neq i$, because coordinating transfers between connections of the same lines is not relevant in practice. However, in the case of the directions of line connections, it may also be the case again that $l = s$ is valid, as it is true that a transfer between different lines may be required in the same direction. The last number in the seven represents the volume of passengers changing at the node $u \in U$ from the connection $k \in P_{il}$ of line $i \in L_u^{**}$ going in the direction of $l \in S$ to a line connection $j \in L_u^{*}$ going in the direction of $s \in S$.

The existence of the coordination requirement in the mathematical model is again defined by the matrix $\boldsymbol{B}$ containing the values 0 or 1. When at node $u \in U$ there is a request to create a coordination link from the incoming line connections $i \in L_u^{**}$ going in the direction of $l \in S$ to the line connections $j \in L_u^{*}$ going in the direction of $s \in S$, then $b_{uiljs} = 1$, otherwise $b_{uiljs} = 0$.

For each pair of lines $i \in L_u^{**}$ and $j \in L_u^{*}$ where $u \in U$, in the situation where $b_{uiljs} = 1$, the value of the transfer time is defined as $tprest_{uij}$ (it is again assumed that the value of the transfer time between the connections of the coordinated lines does not depend on which pair of connections of the given lines is involved). However, in case of passenger volumes f_{uilkjs} transferring (for the selected coordination period) from the connection $k \in P_{il}$ of line $i \in L_u^{**}$ going in the direction of $l \in S$ to a line connection $j \in L_u^{*}$ going in the direction of $s \in S$, it is necessary to proceed differently from the first case described in subsection 6.6. The reason for the different approach is again the existence of a larger number of incoming connections arriving at transfer nodes. As a rule, one of two methods is used.

The first way is again to carry out a detailed transport survey, in which the volumes of passengers arriving on each of the connections of the incoming lines to the transfer nodes will be surveyed, and the directions to which passengers transfer will be monitored in addition to the numbers of passengers arriving on the connections of the incoming lines to the transfer nodes. However, the pitfall of the first approach is that changes in volumes will occur when connections are moved to other time positions, because moving a connection to a different time position may make that connection temporally irrelevant to the passenger. If the results of the directional traffic survey are not available, it is possible to proceed by working with the values of transferring passenger volumes burdened with the maximum level of uncertainty, i.e. the number of passengers transferring between the connections of two lines per coordination period will be distributed evenly between the arriving connections.

For each incoming line $i \in L_u^{**}$, of which the connections serving the coordination node $u \in U$ in the direction of $l \in S$ are being transferred, the earliest possible time the node can be served by the first connection of that line is defined as t_{uil1}, the value of the basic headway is T_i and then the earliest possible departure times of the other connections in the coordination period can be calculated from the relationship using the procedure in Subsection 6.5 . In order to simplify the symbolic notation of the model, let us first denote the possible service times of the node $u \in U$ by the connection $k \in P_{il}$ of line $i \in L_u^{**}$ going in the direction of $l \in S$ with the symbol $\tau_{uilk}(v, w)$.

For each departing line $j \in L_u^{*}$ whose connections serve a coordinating node $u \in U$ in the direction of $s \in S$ and are transferred, the basic headway values are defined as T_j, the earliest possible time for the transfer node to be served by the first outgoing connection of the line is t_{ujs1} and then the earliest possible departure times of the other connections in the coordination period can be calculated from the relation using the procedure in subsection 6.5. In order to simplify the symbolic notation of the model, let us denote the possible times of serving a node $u \in U$ by the connection $p \in P_{js}$ of lines $j \in L_u^{*}$ running in the direction of $s \in S$ with the symbol $\tau_{ujsp}(v, w)$.

The task is to decide on the time shifts of connections in individual directions of the coordinated lines at the respective nodes so that the shifts of connections of individual lines going in the same direction are uniform (to maintain the values of the prescribed headways on the lines) and at the same time to minimize the total time loss of all transferring passengers.

In order to model the decision, we introduce the following variables into the optimization task:

x_{il} non-negative variable modelling the time shift of all connections of the line $i \in L$ in the direction $l \in S$ calculated from their earliest possible time positions,

h_{uilkjs} a non-negative variable modelling the time loss of each passenger transferring at the transfer node $u \in U$ from a connection $k \in P_{il}$ of line $i \in L_u$ travelling in the direction of $l \in S$ to the nearest line connection $j \in L_u$ going in the direction of $s \in S$,

$z_{uilkjsp}$ auxiliary binary variable modelling the formation of the coordination link between the connection $k \in P_{il}$ of the incoming line $i \in L_u$ travelling in the direction of $l \in S$ and the connection $p \in P_{js}$ of the outgoing line $j \in L_u$ going in the direction of $s \in S$ at the transfer node $u \in U$,

v_{il} auxiliary binary variable introduced to increase the value of the base headway by α elementary time units after each even line connection $i \in L_u^{**} \cup L_u^{*}$ in the direction of $l \in S$ within the coordination period serving the last transfer node on the line's route,

w_{il} auxiliary binary variable introduced to increase the value of the base headway by α elementary time units after each odd line connection $i \in L_u^{**} \cup L_u^{*}$ in the direction of $l \in S$ within the coordination period, except for the first connection serving the last transfer node on the line's route.

The symbol M represents the enough high constant. For its value it is possible to select, for example, a number such as 10^6.

The mathematical model of the problem will be of the form (6.52) - (6.62):

$$min\, f(x,h,z,v,w) = \sum_{u \in U} \sum_{i \in L_u^{**}} \sum_{l \in S} \sum_{k \in P_{il}} \sum_{\substack{j \in L_u^{*} \\ j \neq i}} \sum_{s \in S} b_{uiljs} \cdot f_{uilkjs} \cdot h_{uilkjs} \qquad (6.52)$$

subject to:

$$\begin{aligned} &[\tau_{ujsp}(v,w) + x_{js}] - \\ &-[\tau_{uilk}(v,w) + x_{il} + tprest_{uij}] \geq \\ &\geq M \cdot (z_{uilkjsp} - 1) \end{aligned} \qquad \begin{aligned} &\text{for } u \in U,\, i \in L_u^{**},\, j \in L_u^{*}, \\ &j \neq i,\, l \in S,\, s \in S,\, k \in P_{il}, \\ &p \in P_{js} \text{ and } b_{uiljs} = 1 \end{aligned} \qquad (6.53)$$

$$\begin{aligned}&[\tau_{ujsp}(v,w)+x_{js}]- \\ &-[\tau_{uilk}(v,w)+x_{il}+tprest_{uij}]\leq \\ &\leq h_{uilkjs}+M\cdot(1-z_{uilkjsp})\end{aligned} \qquad \begin{aligned}&\text{for } u\in U,\, i\in L_u^{**},\, j\in L_u^{*}, \\ &j\neq i,\, l\in S,\, s\in S,\, k\in P_{il}, \\ &p\in P_{js} \text{ and } b_{uiljs}=1\end{aligned} \qquad (6.54)$$

$$\sum_{p\in P_{js}} z_{uilkjsp}=1 \qquad \begin{aligned}&\text{for } u\in U,\, i\in L_u^{**},\, j\in L_u^{*}, \\ &j\neq i,\, l\in S,\, s\in S,\, k\in P_{il} \\ &\text{and } b_{uiljs}=1\end{aligned} \qquad (6.55)$$

$$v_{il}+w_{il}=1 \qquad \text{for } i\in L_u^{**}\cup L_u^{*} \text{ and } l\in S \qquad (6.56)$$

$$x_{il}\leq a_{il} \qquad \text{for } i\in L_u^{**}\cup L_u^{*} \text{ and } l\in S \qquad (6.57)$$

$$x_{il}\in R_0^+ \qquad \text{for } i\in L_u^{**}\cup L_u^{*} \text{ and } l\in S \qquad (6.58)$$

$$h_{uilkjs}\in R_0^+ \qquad \begin{aligned}&\text{for } u\in U,\, i\in L_u^{**},\, j\in L_u^{*}, \\ &j\neq i,\, l\in S,\, s\in S,\, k\in P_{il} \\ &\text{and } b_{uiljs}=1\end{aligned} \qquad (6.59)$$

$$z_{uilkjsp}\in\{0;1\} \qquad \begin{aligned}&\text{for } u\in U,\, i\in L_u^{**},\, j\in L_u^{*}, \\ &j\neq i,\, l\in S,\, s\in S,\, k\in P_{il}, \\ &p\in P_{js} \text{ and } b_{uiljs}=1\end{aligned} \qquad (6.60)$$

$$v_{il}\in\{0;1\} \qquad \text{for } i\in L_u^{**}\cup L_u^{*} \text{ and } l\in S \qquad (6.61)$$

$$w_{il}\in\{0;1\} \qquad \text{for } i\in L_u^{**}\cup L_u^{*} \text{ and } l\in S \qquad (6.62)$$

Function (6.52) represents the optimization criterion - the total time loss of all transferring passengers at all transfer nodes. The group of constraints (6.53) ensures that in the case of temporal inadmissibility of the positions of connections of coordinated lines running in the directions affected by coordination, no coordination link is created. The group of constraints (6.54) quantifies the time loss of transferring passengers generated by the emergence of coordination links. The group of constraints (6.55) ensures the formation of coordination links. The group of constraints

(6.56) is created only for the set of lines L and their directions in which the alternating headway is applied between the lines (in the case of shuttle lines, it is assumed that when the alternating headway is applied in one direction, it is also applied in the opposite direction). The group of constraints (6.57) shall ensure that any connection time shifts generated to reduce the overall time loss do not exceed the maximum allowable time shift values. The group of constraints (6.58) - (6.62) defines the domains of definition of the variables used in the model.

7 PRACTICAL EXAMPLES OF OPTIMIZATION MODELS AND THEIR SOLUTIONS

Practical examples of the creation of optimization models will be implemented through Example 7.1 - Example 7.3.

<u>Example 7.1</u>

Consider two transfer nodes with implemented coordination served by three lines with a constant headway time of 20 minutes. Line 1 serves both transfer nodes, line 2 serves only transfer node 1 and line 3 serves only transfer node 2 (for the sake of illustration of the solution procedure, we refrain from taking into account other coordination links in other transfer nodes when creating and solving the model in the case of lines 2 and 3). Consider that the travel time of the line 1 connection between the two transfer nodes is 4 minutes. The elementary time unit in terms of time coordination on all lines involved in the coordination is 1 minute. The transfer time between coordinated lines is 5 minutes for transfer node 1 and 6 minutes for transfer node 2.

Further input information is contained in Tables 7.1 - 7.3, based on the forms in Tables 4.1 - 4.3 (Final stops in Table 7.1 are indicated by capital letters of the alphabet, transfer nodes by numbers, transfer node names are omitted in the demonstration example):

Line number	*Direction 1 (Departure stop → Destination stop)*	*Direction 2 (opposite direction if it is a shuttle line) (Departure stop → Destination stop)*
1	*A → 1 → 2 → B*	*B → 2 → 1 → A*
2	*C → 1 → D*	*D → 1 → C*
3	*E → 2 → F*	*F → 2 → E*

Table 7.1: List of coordinated lines and their directions - Example 7.1

For example 1 row 7.1 of Table 7.1 can be interpreted as follows: the route of line 1 in direction 1 starts at the departure stop A, passes through transfer nodes 1 and 2 (in the given order) and ends at the final destination stop B. Line 1 is a shuttle service, so its route in direction 2 starts at the departure stop B, passes through transfer nodes 2 and 1 (in that order) and ends at the final destination stop A.

Name of the transfer node u	Number of the transfer node u	from		for		Transfer time $tprest_{uij}$ [min]
		line number i	direction number l	line number j	direction number s	
	1	1	1	2	1	5
	1	1	1	2	2	5
	2	3	2	1	2	6

Table 7.2: List of coordination nodes with coordinated lines and their directions - Example 7.1

For example 1 row 7.2 of Table 7.2 can be interpreted as follows: at transfer node 1, a coordination link must be ensured between the connections of line 1 going in direction 1 and the connections of line 2 going in direction 1.

Transfer node number u	from			Volume of transferring passengers f_{uilkjs}	for	
	line number i	direction number l	connection number k		line number j	direction number s
1	1	1	1	10	2	1
1	1	1	1	5	2	2
2	3	2	1	12	1	2

Table 7.3: List of coordination links and volumes of transferring passengers using the given coordination link - Example 7.1

The task is to create a mathematical model of the problem, the solution of which will identify the time shifts of line connections in individual directions coordinated at the respective nodes, so that the shifts of connections of individual lines running in the same direction are uniform (to maintain the values of the prescribed headways on the lines), and to minimize the total time loss of all transferring passengers.

We start the creation of the optimization model by calculating the number of connections included in the coordination. On all three coordinated lines a constant headway of 20 minutes is applied, so the coordination period will have the value of $K_p = 20$ *minutes. According to the considerations made in subsection 6.4.1, only 1 connection will be included in the coordination in this case on the incoming lines (line 1 to transfer node 1 in direction 1 and line 3 to transfer node 2 in direction 2), and 2 connections on the outgoing lines (line 2 from transfer node 2 in both directions).*

In the following procedure, we first identify the possible positions of the connections serving each transfer node. Follow the procedure in Example 6.10. Let us construct an analogy of Table 6.1. Since multiple nodes in Example 7.1 are only served by Line 1, Table 7.4 will only contain information for Line 1.

Direction 1, order of served transfer nodes 1 → 2			
Earliest possible service times for transfer node 1 (transfer node in a given direction served first in the sequence)	*-4*	*16* $= t_{1111}$	*36* $= t_{1112}$
The earliest possible service times for transfer node 2 (transfer node in the given direction served as the second in the sequence)	*0* $= t_{2111}$	*20* $= t_{2112}$	---
Direction 2, order of served transfer nodes 2 → 1			
Earliest possible service times for transfer node 2 (transfer node in a given direction served first in the sequence)	*-4*	*16* $= t_{2121}$	*36* $= t_{2122}$
The earliest possible service times for transfer node 1 (transfer node in the given direction served as the second in the sequence)	*0* $= t_{1121}$	*20* $= t_{1122}$	---

Table 7.4: Calculation of the earliest possible service times of transfer nodes 1 and 2

Of all the time data listed in Table 7.4, the following will be part of the coordination task (due to the required coordination links): in the case of transfer node 1, only the value *of the earliest possible service time of connection 1 travelling in direction 1 at time 16 and in the case of transfer node 2, the values of the earliest possible service times of transfer node 2 by connections 1 and 2 travelling in direction 2 at times 16, 36.*

The earliest possible time positions of the service of transfer node 1 of line 2 and transfer node 2 of line 3 are at times 00, 20.

In the next procedure, we identify the values of the incidence matrix ***B***. *For the elements of the matrix, based on the information given in Table 7.2* $b_{11121} = 1$, $b_{11122} = 1$ *and* $b_{23212} = 1$, *for other combinations of indices* $b_{uiljs} = 0$.

In the next procedure we determine the values of the maximum allowed time delays of the coordinated lines. Since only the connections of line 1 in directions 1 and 2, the connections of line 2 in directions 1 and 2 and the connections of line 3 in direction 2 are involved in the network node time coordination, we will allow time shifts and therefore maximum time shifts only for those lines and directions. Therefore, assuming that all time positions can be used within the headway on the lines, the following $a_{11} = 19$, $a_{12} = 19$, $a_{21} = 19$, $a_{22} = 19$ *and* $a_{32} = 19$ *will apply.*

The input variables are then substituted into a mathematical model of the form (only the terms of the objective function and the constraints for the required coordination constraints will be included in the model for illustrative purposes, i.e., when $b_{uiljs} = 1$, *the obligatory constraints will not be listed individually):*

$$min\, f(x,h,z) = 10 \cdot h_{111121} + 5 \cdot h_{111122} + 12 \cdot h_{232112}$$

subject to:

constraints for a coordination link characterised by element $b_{11121} = 1$*:*

$$[0 + x_{21}] - [16 + x_{11} + 5] \geq M \cdot (z_{1111211} - 1)$$

$$[0 + x_{21}] - [16 + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111211})$$

$$[20 + x_{21}] - [16 + x_{11} + 5] \geq M \cdot (z_{1111212} - 1)$$

$$[20 + x_{21}] - [16 + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111212})$$

$$z_{1111211} + z_{1111212} = 1$$

constraints for a coordination link characterised by element $b_{11122} = 1$*:*

$$[0 + x_{22}] - [16 + x_{11} + 5] \geq M \cdot (z_{1111221} - 1)$$

$$[0 + x_{22}] - [16 + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111221})$$
$$[20 + x_{22}] - [16 + x_{11} + 5] \geq M \cdot (z_{1111222} - 1)$$
$$[20 + x_{22}] - [16 + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111222})$$
$$z_{1111221} + z_{1111222} = 1$$

constraints for a coordination link characterised by element $b_{23212} = 1$*:*

$$[16 + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321121} - 1)$$
$$[16 + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321121})$$
$$[36 + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321122} - 1)$$
$$[36 + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321122})$$
$$z_{2321121} + z_{2321122} = 1$$

constraints ensuring that the maximum permitted time delays of connections on coordinated lines are not exceeded:

$$x_{11} \leq 19$$
$$x_{12} \leq 19$$
$$x_{21} \leq 19$$
$$x_{22} \leq 19$$
$$x_{31} \leq 0$$
$$x_{32} \leq 19$$

obligatory constraints:

$x_{il} \in R_0^+$	for $i \in \{1;2;3\}$ and $l \in \{1;2\}$
$h_{uilkjs} \in R_0^+$	for $u \in \{1;2\}$, $i \in \{1;2;3\}$, $j \in \{1;2;3\}$, $j \neq i$, $l \in \{1;2\}$, $s \in \{1;2\}$, $k \in \{1\}$, $b_{uiljs} = 1$
$z_{uilkjsp} \in \{0;1\}$	for $u \in \{1;2\}$, $i \in \{1;2;3\}$, $j \in \{1;2;3\}$, $j \neq i$, $l \in \{1;2\}$,$s \in \{1;2\}$, $k \in \{1\}$,

$$p \in \{1;2\}, b_{uiljs} = 1$$

The universal optimization software Xpress-IVE can be used for solving the constructed model. The text of the programme is as follows:

```
model Example_7_1
uses "mmxprs";
declarations
node=1..2
line=1..3
direction=1..2
connection=1..2
t:array(node,line,direction,connection)of real
a:array(line,direction)of real
x:array(line,direction)of mpvar
z:array(node,line,direction,connection,line,direction,connection)of mpvar
h:array(node,line,direction,connection,line,direction)of mpvar
f:array(node,line,direction,connection,line,direction)of real
trest:array(node,line,line)of real
end-declarations
M:=1000000
(0+x(2,1))-(16+x(1,1)+5)>=M*(z(1,1,1,1,2,1,1)-1)
(0+x(2,1))-(16+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,1))
(20+x(2,1))-(16+x(1,1)+5)>=M*(z(1,1,1,1,2,1,2)-1)
(20+x(2,1))-(16+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,2))
z(1,1,1,1,2,1,1)+z(1,1,1,1,2,1,2)=1
(0+x(2,2))-(16+x(1,1)+5)>=M*(z(1,1,1,1,2,2,1)-1)
(0+x(2,2))-(16+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,1))
(20+x(2,2))-(16+x(1,1)+5)>=M*(z(1,1,1,1,2,2,2)-1)
(20+x(2,2))-(16+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,2))
z(1,1,1,1,2,2,1)+z(1,1,1,1,2,2,2)=1
(16+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,1)-1)
(16+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,1))
(36+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,2)-1)
(36+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,2))
z(2,3,2,1,1,2,1)+z(2,3,2,1,1,2,2)=1
x(1,1)<=19
x(1,2)<=19
x(2,1)<=19
x(2,2)<=19
x(3,2)<=19
z(1,1,1,1,2,1,1)is_binary
z(1,1,1,1,2,1,2)is_binary
z(1,1,1,1,2,2,1)is_binary
z(1,1,1,1,2,2,2)is_binary
```

```
z(2,3,2,1,1,2,1)is_binary
z(2,3,2,1,1,2,2)is_binary
Total_time_loss:=10*h(1,1,1,1,2,1)+5*h(1,1,1,1,2,2)+12*h(2,3,2,1,1,2)
minimize(Total_time_loss)
writeln("Total time loss is: ",getobjval," person-minutes")
writeln("Time shifts on individual lines:")
writeln("x(1,1) = ",getsol(x(1,1))," min")
writeln("x(1,2) = ",getsol(x(1,2))," min")
writeln("x(2,1) = ",getsol(x(2,1))," min")
writeln("x(2,2) = ",getsol(x(2,2))," min")
writeln("x(3,2) = ",getsol(x(3,2))," min")
writeln("Waiting:")
writeln("h(1,1,1,1,2,1) = ",getsol(h(1,1,1,1,2,1))," min")
writeln("h(1,1,1,1,2,2) = ",getsol(h(1,1,1,1,2,2))," min")
writeln("h(2,3,2,1,1,2) = ",getsol(h(2,3,2,1,1,2))," min")
writeln("Coordination links:")
writeln("z(1,1,1,1,2,1,1) = ",getsol(z(1,1,1,1,2,1,1)))
writeln("z(1,1,1,1,2,1,2) = ",getsol(z(1,1,1,1,2,1,2)))
writeln("z(1,1,1,1,2,2,1) = ",getsol(z(1,1,1,1,2,2,1)))
writeln("z(1,1,1,1,2,2,2) = ",getsol(z(1,1,1,1,2,2,2)))
writeln("z(2,3,2,1,1,2,1) = ",getsol(z(2,3,2,1,1,2,1)))
writeln("z(2,3,2,1,1,2,2) = ",getsol(z(2,3,2,1,1,2,2)))
end-model
```

After the optimization calculation was completed, the following results were obtained:

```
Total time loss is: 0 person-minutes
Time shifts on individual lines:
x(1,1) = 0 min
x(1,2) = 9 min
x(2,1) = 1 min
x(2,2) = 1 min
x(3,2) = 19 min
Waiting:
h(1,1,1,1,2,1) = 0 min
h(1,1,1,1,2,2) = 0 min
h(2,3,2,1,1,2) = 0 min
Coordination links:
z(1,1,1,1,2,1,1) = 0
z(1,1,1,1,2,1,2) = 1
z(1,1,1,1,2,2,1) = 0
z(1,1,1,1,2,2,2) = 1
```

z(2,3,2,1,1,2,1) = 1
z(2,3,2,1,1,2,2) = 0

The results can be interpreted as follows:

Variable values x_{il} indicate the time shifts of the connections of the coordinated lines. In order to achieve the optimal solution (total time loss of 0 person-minutes), the following time shifts of the connections included in the network node time coordination must be performed:

In the listing of the results of the optimization calculation it is stated $x_{11} = 0$ minutes. Variable x_{11} models the time shift of the connections of line 1 in direction 1 relative to their earliest possible time positions. Line 1 in direction 1 acts as an incoming line to transfer node 1, so only one connection serving transfer node 1 in that direction is included in the coordination period, given the input conditions of the task. However, just to illustrate the whole situation, the same time shift will also be applied to other connections of the given line going in a given direction and serving both transfer nodes. The earliest possible service of transfer node 1 by a connection 1 of line 1 going in direction 1 occurs at time 16 (additional services at times 36 and 56) and then at transfer node 2 at time 00 (additional services at times 20 and 40). This means that after applying the time shift of $x_{11} = 0$ minutes, operation at transfer node 1 by connection 1 of line 1 going in direction 1 will remain at time 16 (other connections will serve the transfer node at times 36 and 56) and the operation at transfer node 2 by connection 1 of line 1 going in direction 1 will remain at time 00 (other connections will serve the transfer node at times 20, 40).

The results of the optimization calculation also state that $x_{12} = 9$ minutes. Variable x_{12} models the time shift of the connections of line 1 in direction 2 relative to their earliest possible time positions. Line 1 in direction 2 acts as the outgoing line from transfer node 2, so the coordination period includes two connections serving transfer node 2 in the given direction, according to the input conditions of the task. Of course, analogous to the previous case, the same time shift will also apply to other connections of the line in question going in a given direction and serving both transfer nodes. The earliest *possible service of transfer node 1 by connections 1 and 2 of line 1 going in direction 2 occurs at times 00, 20 (next service at time 40) and of transfer node 2 at times 16, 36 (next service at time 56). This means that after applying the time shift of* $x_{12} = 9$ *minutes, the service of transfer node 1 by connections 1 and 2 of line 1 going in direction 2 will occur at times 09, 29 (next service at time 49) and then transfer node 2 at times 25, 45 (next service at time 05 of the following hour).*

The results of the optimization calculation also state that $x_{21} = 1$ *minutes. Variable* x_{21} *models the time shift of the connections of line 2 in direction 1 relative to their earliest possible time positions. Line 2 in direction 1 acts as the outgoing line from transfer node 1, so the coordination period includes two connections serving transfer node 1 in the given direction, according to the input conditions of the task. Of course, analogous to the previous cases, the same time shift will also apply to other connections of the given line running in the same direction. The earliest possible service of transfer node 1 by connections 1 and 2 of line 2 going in direction 1 occurs at times 00, 20 (next service at time 40) This means that after applying the time shift of* $x_{21} = 1$ *minutes, the service at transfer node 1 by connections 1 and 2 of line 2 going in direction 1 occurs at times 01, 21 (next service at time 41).*

The results of the optimization calculation also state that $x_{22} = 1$ *minutes. Variable* x_{22} *models the time shift of the connections of line 2 in direction 2 relative to their earliest possible time positions. Line 2 in direction 2 is the outgoing line from transfer node 1, so the coordination period includes two connections serving transfer node 1 in the given direction, according to the input conditions of the task. Of course, analogous to the previous cases, the same time shift will also apply to other connections of the given line running in the same direction. The earliest possible service of the transfer node 1 by connections 1 and 2 of line 2 going in direction 2 occurs again at times 00, 20 (the next service again at time 40) This means that after the application of the time shift of* $x_{22} = 1$ *minutes, the service of transfer node 1 by connections 1 and 2 of line 2 going in direction 2 occurs at times 01, 21 (next service at time 41).*

The results of the optimization calculation also state that $x_{32} = 19$ *minutes. Variable* x_{32} *models the time shift of the connections of line 3 in direction 2 relative to their* *earliest possible time positions. Line 3 in direction 2 acts as an incoming line to transfer node 2, so only one connection serving transfer node 2 in that direction is included in the coordination period, according to the input conditions of the task. Of course, analogous to the previous cases, the same time shift will also apply to other connections of the given line running in the same direction. The earliest possible service of transfer node 2 by connection 1 of line 3 going in direction 2 occurs again at time 00 (other services again at times 20 and 40) This means that after the application of the time shift of* $x_{32} = 19$ *minutes, the service of transfer node 2 by connection 1 of line 3 going in direction 2 occurs at time 19 (further services at times 39 and 59).*

In the example, 3 coordination links were defined: in transfer node 1 from line 1 connections going in direction 1 to line 2 connections going in direction 1 and to line 2 connections going in direction 2 and at transfer node 2 from line 3 connections going in direction 2 to line 1 connections going in direction 2. According to the output of the results, transferring passengers should not incur time loss, which we check via the values of the variables h_{uilkjs} *or their sum.*

Connection 1 of incoming line 1 going in direction 1 serves transfer node 1 at time 16. The transfer time between connection 1 of incoming line 1 going in direction 1 and connections of outgoing line 2 going in direction 1 is 5 minutes. Passengers arriving by connection 1 of line 1, direction 1, at 16 will therefore be ready for the departure of line 2, direction 1, at 21. Connections of line 2 to direction 1 depart at times 01, 21, and since the time when passengers transferring from connection 1 of line 1 going in direction 1 are ready to depart with connections of line 2 travelling in direction 1 coincides with the time position of the service of transfer node 1 by connection 2 of line 2 going in direction 1, then also the time loss of all transferring passengers in the coordination link in question will be 0 person-minutes.

The same situation arises in the case of a transfer from connection 1 of line 1 going in direction 1 to connection 2 of line 2 going in direction 2. Connection 1 of incoming line 1 going in direction 1 serves transfer node 1 at time 16. The transfer time between connection 1 of incoming line 1 going in direction 1 and connections of outgoing line 2 going in direction 2 is 5 minutes. Passengers arriving by connection 1, line 1, going in *direction 1 at 16 will therefore be ready for the departure of connection 2, line 2, going in direction 2 at 21. Connections of line 2 to direction 2 depart at times 01, 21, and since the time at which passengers transferring from connection 1 of line 1 going in direction 1 are ready to depart by connections of line 2 going in direction 2 coincides with the time position of the service of transfer node 1 by connection 2 of line 2 going in direction 2, then also the time loss of all transferring passengers in the coordination link in question will be 0 person-minutes.*

Next, let us check the value of the time loss of transferring passengers for the third coordination link. Connection 1 of incoming line 3 going in direction 2 serves transfer node 2 at time 19. The transfer time between connection 1 of incoming line 3 going in direction 2 and connections of outgoing line 1 going in direction 2 is 6 minutes. Passengers arriving by connection 1, line 3, going in direction 2 at 19 will therefore be ready for the departure of a connection of line 1, in direction 2, at 25. Line 1 connections to direction 2 depart at times 25, 45, and since the time when passengers transferring from connection 1 of line 3 travelling in direction 2 are ready to depart with connection 1 of line 3 travelling in direction 2 coincides with the time position of the service of transfer node 2 by connection 1 of line 1 travelling in direction 2, then also the time loss of all transferring passengers in the given coordination link will be 0 person-minutes.

Since the values of the time loss of the transferring passengers within all three coordination links are equal to 0, the total time loss is also equal to 0.

At the end of the interpretation of the example results, the values of the variables $z_{uilkjsp}$ will be interpreted. From the point of view of the results of the optimization calculation, the values of variable equal to 1 listed above are important, as they represent the formation of the transfer links. For the results of Example 7.2 $z_{1111212} = 1$, $z_{1111222} = 1$ and $z_{2321121} = 1$ are valid. The values of the variables are intended to confirm the interpretation of the results made in the previous paragraphs.

Variable $z_{1111212}$ and its value 1 represents the formation of a coordination link at the transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 1 (fourth index) and line 2 (fifth index), direction 1 (sixth index) and its *connection 2 (seventh index). Variable $z_{1111222}$ and its value 1 represents the formation of a coordination link at the transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 1 (fourth index) and line 2 (fifth index), direction 2 (sixth index) and its connection 2 (seventh index). Variable $z_{2321121}$ and its value 1 represents the formation of a coordination link at transfer node 2 (first index) between line 3 (second index), direction 2 (third index) and its connection 1 (fourth index) and line 1 (fifth index), direction 2 (sixth index) and its connection 1 (seventh index).*

Example 7.2

Consider again two transfer nodes with implemented coordination served by three lines, of which lines 1 and 3 have a 20-minute headway and line 2 has a 10-minute headway. The line routing is the same as in Example 7.1, i.e., line 1 serves both transfer nodes, line 2 serves only transfer node 1 and line 3 serves only transfer node 2 (by analogy, to illustrate the solution procedure in the case of lines 2 and 3, we refrain from taking into account other coordination links in other transfer nodes when creating and solving the model). Consider that the travel time of a line 1 connection between the two transfer nodes is 12 minutes. The elementary time unit in terms of time coordination on all lines involved in the coordination is 1 minute. The transfer time between coordinated lines is 5 minutes for transfer node 1 and 6 minutes for transfer node 2.

Further input information is contained in Tables 7.5 - 7.7, based on the forms in Tables 4.1 - 4.3 (the final stops in Table 7.5 are indicated by capital letters of the alphabet, the transfer nodes by numbers, the names of the transfer nodes are omitted in the demonstration example):

Line number	Direction 1 (Departure stop → Destination stop)	Direction 2 (opposite direction if it is a shuttle line) (Departure stop → Destination stop)
1	A → 1 → 2 → B	B → 2 → 1 → A
2	C → 1 → D	D → 1 → C
3	E → 2 → F	F → 2 → E

Table 7.5: List of coordinated lines and their directions - Example 7.2

Name of the transfer node u	Transfer node number u	from		for		Transfer time $tprest_{uij}$ [min]
		line number i	direction number l	line number j	direction number s	
	1	1	1	2	1	5
	1	1	1	2	2	5
	2	3	2	1	2	6

Table 7.6: List of coordination nodes with coordinated lines and their directions - Example 7.2

Transfer node number u	from			Volume of transferring passengers f_{uilkjs}	for	
	line number i	direction number l	connection number k		line number j	direction number s
1	1	1	1	1	2	1
1	1	1	1	1	2	2
2	3	2	1	1	1	2

Table 7.7: List of coordination links and volumes of transferring passengers using the given coordination link - Example 7.2

The volumes of transferring passengers are chosen to be unit, i.e., all required coordination links have the same importance (weight), which corresponds to the maximum level of uncertainty.

The task is to create a mathematical model of the problem, the solution of which will identify the time shifts of line connections in individual directions coordinated at the respective nodes so that the shifts of connections of individual lines going in the same direction are uniform (to maintain the values of the prescribed headways on the lines) and to minimize the total time loss of all transferring passengers.

We start the creation of the optimization model by calculating the number of connections included in the coordination. On two coordinated lines a constant headway of 20 minutes is applied and on one coordinated line a constant headway of 10 minutes is applied. Thus, the coordination period will have the value of $K_p = 20$ *minutes, which corresponds to the smallest common multiple of the values of the constant headways of 10 and 20 minutes. According to the considerations made in subsection 6.4.1, the coordination task in this case will include only 1 connection on the incoming line 1 to transfer node 1 in direction 1, 2 connections on the outgoing line 1 from transfer node 2 in direction 2 and 3 connections on the outgoing line 2 from transfer node 2 in directions 1 and 2.*

In the following procedure, we first identify the possible time positions of the connections serving each transfer node. Follow the procedure in Example 6.10. Let us construct an analogy of Table 6.1. Since multiple transfer nodes in Example 7.2 are served only by Line 1, Table 7.8 will only contain information for Line 1.

Direction 1, order of served transfer nodes 1 → 2			
Earliest possible service times for transfer node 1 (transfer node in a given direction served first in the sequence)	-12	08 $= t_{1111}$	28 $= t_{1112}$
The earliest possible service times for transfer node 2 (transfer node in the given direction served as the second in the sequence)	00 $= t_{2111}$	20 $= t_{2112}$	---
Direction 2, order of served transfer nodes 2 → 1			
Earliest possible service times for transfer node 2 (transfer node in a given direction served first in the sequence)	-12	08 $= t_{2121}$	28 $= t_{2122}$
The earliest possible service times for transfer node 1 (transfer node in the given direction served as the second in the sequence)	00 $= t_{1121}$	20 $= t_{1122}$	---

Table 7.8: Calculation of the earliest possible service times of transfer nodes 1 and 2

Of all the time data listed in Table 7.8, the following will be part of the coordination task (due to the required coordination links): in the case of transfer node 1, only the value of the earliest possible time of connection 1 going in direction 1 at time 08 and in the case of transfer node 2, the values of the earliest possible service times of transfer node 2 by connections 1 and 2 going in direction 2 at times 08, 28.

The earliest possible time positions at transfer node 1 of the connection of the outgoing line 2 in both directions occur in the coordination period at times 00, 10, 20 and of transfer node 2 of the connection of incoming line 3 in the coordination period at time 00.

*In the next procedure, we again identify the values of the incidence matrix **B**. Since the same coordination links are defined as in Example 7.1, the information given in Table 7.2 applies to the elements of the matrix $b_{11121} = 1$, $b_{11122} = 1$ and $b_{23212} = 1$, for the other index combinations $b_{uiljs} = 0$ is valid.*

In the next procedure we determine the values of the maximum allowed time shifts of the coordinated lines. Since only the connections of line 1 in directions 1 and 2, the connections of line 2 in directions 1 and 2 and the connections of line 3 in direction 2 are involved in the network node time coordination, we will allow time shifts and therefore maximum time shifts only for those lines and directions. Therefore, assuming that all time positions can be used within the headway on the lines, the following $a_{11} = 19$, $a_{12} = 19$, $a_{21} = 9$, $a_{22} = 9$ and $a_{32} = 19$ will apply.

The input variables are then substituted into a mathematical model of the form (only the terms of the objective function and the constraints for the required coordination links will be included in the model for illustrative purposes, i.e., when $b_{uiljs} = 1$, the obligatory constraints will be partially written in abbreviated form):

$$min\, f(x, h, z) = 1 \cdot h_{111121} + 1 \cdot h_{111122} + 1 \cdot h_{232112}$$

subject to:

constraints for a coordination link characterised by an element $b_{11121} = 1$:

$$[0 + x_{21}] - [8 + x_{11} + 5] \geq M \cdot (z_{1111211} - 1)$$
$$[0 + x_{21}] - [8 + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111211})$$
$$[10 + x_{21}] - [8 + x_{11} + 5] \geq M \cdot (z_{1111212} - 1)$$
$$[10 + x_{21}] - [8 + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111212})$$
$$[20 + x_{21}] - [8 + x_{11} + 5] \geq M \cdot (z_{1111213} - 1)$$
$$[20 + x_{21}] - [8 + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111213})$$
$$z_{1111211} + z_{1111212} + z_{1111213} = 1$$

constraints for a coordination link characterised by an element $b_{11122} = 1$:

$$[0 + x_{22}] - [8 + x_{11} + 5] \geq M \cdot (z_{1111221} - 1)$$
$$[0 + x_{22}] - [8 + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111221})$$
$$[10 + x_{22}] - [8 + x_{11} + 5] \geq M \cdot (z_{1111222} - 1)$$
$$[10 + x_{22}] - [8 + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111222})$$
$$[20 + x_{22}] - [8 + x_{11} + 5] \geq M \cdot (z_{1111223} - 1)$$
$$[20 + x_{22}] - [8 + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111223})$$
$$z_{1111221} + z_{1111222} + z_{1111223} = 1$$

constraints for a coordination link characterised by an element $b_{23212} = 1$:

$$[8 + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321121} - 1)$$
$$[8 + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321121})$$
$$[28 + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321122} - 1)$$
$$[28 + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321122})$$
$$z_{2321121} + z_{2321122} = 1$$

constraints ensuring that the maximum permitted time shifts of connections on coordinated lines are not exceeded:

$$x_{11} \leq 19$$
$$x_{12} \leq 19$$
$$x_{21} \leq 9$$
$$x_{22} \leq 9$$
$$x_{31} \leq 0$$
$$x_{32} \leq 19$$

obligatory constraints:

$$x_{il} \in R_0^+ \quad \text{for } i \in \{1;2;3\} \text{ and } l \in \{1;2\}$$

$$h_{111121} \in R_0^+ \qquad h_{111122} \in R_0^+ \qquad h_{232112} \in R_0^+$$
$$z_{1111211} \in \{0;1\} \qquad z_{1111212} \in \{0;1\} \qquad z_{1111213} \in \{0;1\}$$
$$z_{1111221} \in \{0;1\} \qquad z_{1111222} \in \{0;1\} \qquad z_{1111223} \in \{0;1\}$$
$$z_{2321121} \in \{0;1\} \qquad z_{2321122} \in \{0;1\}$$

The program text that can be solved in the Xpress-IVE optimization software is as follows:

```
model Example_7_2
uses "mmxprs";
declarations
node=1..2
line=1..3
direction=1..2
connection=1..3
t:array(node,line,direction,connection)of real
a:array(line,direction)of real
x:array(line,direction)of mpvar
z:array(node,line,direction,connection,line,direction,connection)of mpvar
h:array(node,line,direction,connection,line,direction)of mpvar
```

```
f:array(node,line,direction,connection,line,direction)of real
trest:array(node,line,line)of real
end-declarations
M:=1000000
(0+x(2,1))-(8+x(1,1)+5)>=M*(z(1,1,1,1,2,1,1)-1)
(0+x(2,1))-(8+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,1))
(10+x(2,1))-(8+x(1,1)+5)>=M*(z(1,1,1,1,2,1,2)-1)
(10+x(2,1))-(8+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,2))
(20+x(2,1))-(8+x(1,1)+5)>=M*(z(1,1,1,1,2,1,2)-1)
(20+x(2,1))-(8+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,3))
z(1,1,1,1,2,1,1)+z(1,1,1,1,2,1,2)+z(1,1,1,1,2,1,3)=1
(0+x(2,2))-(8+x(1,1)+5)>=M*(z(1,1,1,1,2,2,1)-1)
(0+x(2,2))-(8+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,1))
(10+x(2,2))-(8+x(1,1)+5)>=M*(z(1,1,1,1,2,2,2)-1)
(10+x(2,2))-(8+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,2))
(20+x(2,2))-(8+x(1,1)+5)>=M*(z(1,1,1,1,2,2,3)-1)
(20+x(2,2))-(8+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,3))
z(1,1,1,1,2,2,1)+z(1,1,1,1,2,2,2)+z(1,1,1,1,2,2,3)=1
(8+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,1)-1)
(8+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,1))
(28+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,2)-1)
(28+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,2))
z(2,3,2,1,1,2,1)+z(2,3,2,1,1,2,2)=1
x(1,1)<=19
x(1,2)<=19
x(2,1)<=9
x(2,2)<=9
x(3,2)<=19
z(1,1,1,1,2,1,1)is_binary
z(1,1,1,1,2,1,2)is_binary
z(1,1,1,1,2,1,3)is_binary
z(1,1,1,1,2,2,1)is_binary
z(1,1,1,1,2,2,2)is_binary
z(1,1,1,1,2,2,3)is_binary
z(2,3,2,1,1,2,1)is_binary
z(2,3,2,1,1,2,2)is_binary
Total_time_loss:=1*h(1,1,1,1,2,1)+1*h(1,1,1,1,2,2)+1*h(2,3,2,1,1,2)
minimize(Total_time_loss)
writeln("Total time loss is: ",getobjval," person-minutes")
writeln("Time shifts of individual lines:")
writeln("x(1,1) = ",getsol(x(1,1))," min")
writeln("x(1,2) = ",getsol(x(1,2))," min")
writeln("x(2,1) = ",getsol(x(2,1))," min")
writeln("x(2,2) = ",getsol(x(2,2))," min")
writeln("x(3,1) = ",getsol(x(3,1))," min")
writeln("x(3,2) = ",getsol(x(3,2))," min")
writeln("Waiting:")
writeln("h(1,1,1,1,2,1) = ",getsol(h(1,1,1,1,2,1))," min")
```

```
writeln("h(1,1,1,1,2,2) = ",getsol(h(1,1,1,1,2,2))," min")
writeln("h(2,3,2,1,1,2) = ",getsol(h(2,3,2,1,1,2))," min")
writeln("Coordination links:")
writeln("z(1,1,1,1,2,1,1) = ",getsol(z(1,1,1,1,2,1,1)))
writeln("z(1,1,1,1,2,1,2) = ",getsol(z(1,1,1,1,2,1,2)))
writeln("z(1,1,1,1,2,1,3) = ",getsol(z(1,1,1,1,2,1,3)))
writeln("z(1,1,1,1,2,2,1) = ",getsol(z(1,1,1,1,2,2,1)))
writeln("z(1,1,1,1,2,2,2) = ",getsol(z(1,1,1,1,2,2,2)))
writeln("z(1,1,1,1,2,2,3) = ",getsol(z(1,1,1,1,2,2,3)))
writeln("z(2,3,2,1,1,2,1) = ",getsol(z(2,3,2,1,1,2,1)))
writeln("z(2,3,2,1,1,2,2) = ",getsol(z(2,3,2,1,1,2,2)))
end-model
```

After the optimization calculation was completed, the following results were obtained:

```
Total time loss is: 0 person-minutes
Time shifts of individual lines:
x(1,1) = 0 min
x(1,2) = 17 min
x(2,1) = 3 min
x(2,2) = 3 min
x(3,1) = 0 min
x(3,2) = 19 min
Waiting:
h(1,1,1,1,2,1) = 0 min
h(1,1,1,1,2,2) = 0 min
h(2,3,2,1,1,2) = 0 min
Coordination links:
z(1,1,1,1,2,1,1) = 0
z(1,1,1,1,2,1,2) = 1
z(1,1,1,1,2,1,3) = 0
z(1,1,1,1,2,2,1) = 0
z(1,1,1,1,2,2,2) = 1
z(1,1,1,1,2,2,3) = 0
z(2,3,2,1,1,2,1) = 1
z(2,3,2,1,1,2,2) = 0
```

The results can be interpreted as follows:

Variable values x_{il} indicate the time shift of the connections of the coordinated lines. In order to achieve the optimal solution (total time loss of 0 person-minutes), the following time shifts of the connections included in the network node time coordination must be performed:

In the listing of the results of the optimization calculation it is stated $x_{11} = 0$ minutes. Variable x_{11} models the time shift of the connections of line 1 in direction 1 relative to their earliest possible time positions. Line 1 in direction 1 acts as an incoming line to transfer node 1, so only one connection serving transfer node 1 in that direction is included in the coordination period, according to the input conditions of the task. Naturally, other connections of the line running in the same direction and serving both transfer nodes will be subject to the same time delay. The earliest possible service of transfer node 1 by connection 1 of line 1 going in direction 1 occurs at time 08 (additional connections at times 28 and 48) and then at transfer node 2 at time 00 (additional connections at times 20 and 40). This means that after applying the time shift of $x_{11} = 0$ minutes, the service of transfer node 1 by connection 1 of line 1 going in direction 1 will remain at time 08 (further connections will serve the transfer node at times 28 and 48) and the service of transfer node 2 by connection 1 of line 1 going in direction 1 will remain at time 00 (further connections will serve the transfer node at times 20, 40).

The results of the optimization calculation also state that $x_{12} = 17$ minutes. Variable x_{12} models the time shift of the connections of line 1 in direction 2 relative to their earliest possible time positions. Line 1 in direction 2 acts as the outgoing line from transfer node 2, so the coordination period includes two connections serving transfer node 2 in the given direction, according to the input conditions of the task. Of course, analogous to the previous case, the same time shift will also apply to other connections of the line in question going in a given direction and serving both transfer nodes. The earliest possible service of transfer node 1 by connections 1 and 2 of line 1 going in direction 2 occurs at times 00, 20 (next connection at time 40) and of transfer node 2 at times 08, 28 (next connection at time 48). This means that after applying the time shift of $x_{12} = 17$ minutes, the service of transfer node 1 by connections 1 and 2 of line 1 travelling in direction 2 will occur at times 17, 37 (next connection at time 57) and the service of transfer node 2 then at times 25, 45 (next connection at time 05 of the following hour).

The results of the optimization calculation also state that $x_{21} = 3$ minutes. Variable x_{21} models the time shift of the connections of line 2 in direction 1 relative to their earliest *possible time positions. Line 2 in direction 1 acts as the outgoing line from transfer node 1, so the coordination period includes three connections serving transfer node 1 in the given direction, according to the input conditions of the task. Of course, analogous to the previous case, the same time shift will also apply to other connections of the line in question going in the given direction and interval. The earliest possible service of transfer node 1 by connections 1, 2 and 3 of line 2 going in direction 1 occurs at times 00, 10, 20 (other connections at times 30, 40, 50) This means that after the application of the time shift of* $x_{21} = 3$ *minutes, the service of the transfer node 1 with connections 1, 2 and 3 of line 2 going in direction 1 occurs at times 03, 13, 23 (next connections at times 33, 43, 53).*

The results of the optimization calculation also state that $x_{22} = 3$ *minutes. Variable* x_{22} *models the time shift of the connections of line 2 in direction 2 relative to their earliest possible time positions. Line 2 in direction 2 acts as the outgoing line from transfer node 1, so the coordination period includes three connections serving transfer node 1 in the given direction, according to the input conditions of the task. Of course, analogous to the previous case, the same time shift will also apply to other connections of the line in question going in the given direction. The earliest possible service of transfer node 1 by connections 1, 2 and 3 of line 2 going in direction 2 occurs at times 00, 10, 20 (other connections at times 30, 40, 50) This means that after the application of the time shift of* $x_{22} = 3$ *minutes, the service of the transfer node 1 with connections 1, 2 and 3 of line 2 going in direction 2 occurs at times 03, 13, 23 (next connections at times 33, 43, 53).*

The results of the optimization calculation also state that $x_{32} = 19$ *minutes. Variable* x_{32} *models the time shift of the connections of line 3 in direction 2 relative to their earliest possible time positions. Line 3 in direction 2 acts as an incoming line to transfer node 2, so only one connection serving transfer node 2 in that direction is included in the coordination period, according to the input conditions of the task. Of course, analogous to the previous cases, the same time shift will also be applied to other connections of the given line running in a given direction. The earliest possible service of transfer node 2 by connection 1 of line 3 going in direction 2 occurs again at time 00 (other connections again at times 20 and 40) This means that after the application of* *the time shift of $x_{32} = 19$ minutes, the service of transfer node 2 by connection 1 of line 3 going in direction 2 occurs at time 19 (further connections at times 39 and 59).*

In the example, 3 coordination links were defined: at transfer node 1 from line 1 connections going in direction 1 to line 2 connections going in direction 1 and to line 2 connections going in direction 2 and at transfer node 2 from line 3 connections going in direction 2 to line 1 connections going in direction 2. According to the output of the results, transferring passengers should not incur time losses, which we check in via the values of the variables h_{uilkjs}or their sum.

Connection 1 of incoming line 1 going in direction 1 serves transfer node 1 at time 08. The transfer time between connection 1 of incoming line 1 going in direction 1 and connections of outgoing line 2 going in direction 1 is 5 minutes. Passengers arriving by connection 1 of line 1 in direction 1 at 08 will therefore be ready for the departure of line 2, direction 1, at 13. Connections of line 2 to direction 1 depart at times 03, 13, 23, and since the time when passengers transferring from connection 1 of line 1 going in direction 1 are ready to depart by connections of line 2 going in direction 1 coincides with the time position of the service of transfer node 1 by connection 2 of line 2 going in direction 1, then also the time loss of all transferring passengers in a given coordination link will be 0 person-minutes.

The same situation arises in the case of a transfer from connection 1 of line 1 going in direction 1 to connection 2 of line 2 going in direction 2. Connection 1 of incoming line 1 going in direction 1 serves transfer node 1 at time 08. The transfer time between connection 1 of incoming line 1 going in direction 1 and connections of outgoing line 2 going in direction 2 is 5 minutes. Passengers arriving by connection 1 of line 1 going in direction 1 at 08 will therefore be ready for the departure of connection 2 of line 2 going in direction 2 at 13. Connections of line 2 to direction 2 depart at times 03, 13, 23, and since the time when passengers transferring from connection 1 of line 1 going in direction 1 are ready to depart by connections of line 2 going in direction 2 coincides with the time position of the service of transfer node 1 by connection 2 of line 2 going in direction 2, then also the time loss of all transferring passengers in the given coordination link will be 0 person-minutes.

Next, let us check the value of the time loss of transferring passengers for the third coordination link. Connection 1 of incoming line 3 going in direction 2 serves transfer node 2 at time 19. The transfer time between connection 1 of incoming line 3 going in direction 2 and connections of outgoing line 1 going in direction 2 is 6 minutes. Passengers arriving by service 1 of line 3 going in direction 2 at 19 will therefore be ready for the departure of a connection of line 1 in direction 2 at 25. Line 1 connections to direction 2 depart at times 25, 45, and since the time when passengers transferring from connection 1 of line 3 travelling in direction 2 are ready to depart with line 1 connections travelling in direction 2 coincides with the time position of the service of transfer node 2 by connection 1 of line 1 travelling in direction 2, then also the time loss of all transferring passengers in the given coordination link will be 0 person-minutes.

Since the values of the time loss of the transferring passengers within all three coordination links are equal to 0, the total time loss is also equal to 0.

At the end of the interpretation of the example results, the values of the variables $z_{uilkjsp}$ *will be interpreted. From the point of view of the results of the optimization calculation, the values of variable 1 listed above are important, as they represent the formation of the transfer links. For the results of Example 7.2* $z_{1111212} = 1$, $z_{1111222} = 1$ *and* $z_{2321121} = 1$ *are valid. The values of the variables are intended to confirm the interpretation of the results made in the previous paragraphs.*

Variable $z_{1111212}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 1 (fourth index) and line 2 (fifth index), direction 1 (sixth index) and its connection 2 (seventh index). Variable* $z_{1111222}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 1 (fourth index) and line 2 (fifth index), direction 2 (sixth index) and its connection 2 (seventh index). Variable* $z_{2321121}$ *and its value 1 represents the formation of a coordination link at transfer node 2 (first index) between line 3 (second index), direction 2 (third index) and its connection 1 (fourth* *index) and line 1 (fifth index), direction 2 (sixth index) and its connection 1 (seventh index).*

<u>Example 7.3 - Combination of operational variants with alternating headway on arrival to and departure from transfer nodes</u>

Let us consider two transfer nodes with implemented coordination served by one line with alternating headways of 7 and 8 minutes and two lines with a constant headway of 10 minutes. Line 1 with the alternating headway serves both transfer nodes, line 2 with a constant headway of 10 minutes serves only transfer node 1 and line 3 with a constant headway of 10 minutes serves only transfer node 2 (for the sake of illustration of the solution procedure, again in the creation and solution of the model in the case of lines 2 and 3 we refrain from taking into account other coordination links in other transfer nodes). Consider that the travel time of a line 1 connections between the two transfer nodes is 12 minutes. The elementary time unit in terms of time coordination on all lines involved in the coordination is 1 minute. The transfer time between the coordinated lines is again 5 minutes for transfer node 1 and 6 minutes for transfer node 2.

Further input information is contained in Tables 7.9 - 7.11, based on the forms in Tables 4.1 - 4.3 (the end stops in Table 7.9 are indicated by capital letters of the alphabet, the transfer nodes by numbers, the names of the transfer nodes are omitted in the demonstration example):

Line number	*Direction 1 (Departure stop → Destination stop)*	*Direction 2 (opposite direction if it is a shuttle line) (Departure stop → Destination stop)*
1	*A → 1 → 2 → B*	*B → 2 → 1 → A*
2	*C → 1 → D*	*D → 1 → C*
3	*E → 2 → F*	*F → 2 → E*

Table 7.9: List of coordinated lines and their directions - Example 7.3

Name of the transfer node u	Number of the transfer node u	from		for		Transfer time $tprest_{uij}$ [min]
		line number i	direction number l	line number j	direction number s	
	1	1	1	2	1	5
	1	1	1	2	2	5
	2	3	2	1	2	6

Table 7.10: List of coordination nodes with coordinated lines and their directions - Example 7.3

The list of coordination links shown in Table 7.11 is more extensive than in the previous examples, which results from the observation that on the alternating headway of line 1, 4 connections are included in the coordination period. The average numbers of transferring passengers are chosen to be a unit, i.e., all coordination links are assumed to be of equal importance, corresponding to the maximum level of uncertainty.

The task is to create a mathematical model of the problem, the solution of which will identify the time shifts of line connections in individual directions coordinated at the respective nodes, so that the shifts of connections of individual lines running in the same direction are uniform (to maintain the values of the prescribed headways on the lines), and to minimize the total time loss of all transferring passengers.

Transfer node number u	from			Volume of transferring passengers f_{uilkjs}	for	
	line number i	direction number l	connection number k		line number j	direction number s
1	1	1	1	1	2	1
1	1	1	2	1	2	1
1	1	1	3	1	2	1
1	1	1	4	1	2	1
1	1	1	1	1	2	2
1	1	1	2	1	2	2
1	1	1	3	1	2	2
1	1	1	4	1	2	2
2	3	2	1	1	1	2
2	3	2	2	1	1	2
2	3	2	3	1	1	2

Table 7.11: List of coordination links and volumes of transferring passengers using the given coordination link - Example 7.3

We start the creation of the optimization model by calculating the number of connections included in the coordination. On one line there is an alternating headway with a base headway value of 7 minutes, on the other two lines there are constant headways of 10 minutes, so the coordination period will have a value of $K_p = 30$ minutes. According to the considerations made in subsection 6.4.1, the coordination task will consider 4 connections in the case of line 1 arriving at transfer node 1, 5 connections in the case of line 1 departing from transfer node 2, 4 connections in both directions in the case of line 2 departing from transfer node 2 and 3 connections in the case of line 3 arriving at transfer node 2.

In the following procedure, we first identify the possible positions of the connections serving each transfer node. Follow the procedure in Example 6.10. Let us construct an analogy of Table 6.1. Since multiple nodes in Example 7.3 are served by line 1 only, Table 7.12 will contain only information for line 1, the earliest possible positions will be *based on the basic headway value increased by the values of the binary variables v_{1l} and w_{1l}, where $l = 1{,}2$.*

Direction 1, order of served transfer nodes 1 → 2						
Earliest possible service times for transfer node 1	-12	$-5 + +v_{11}$	$2 + +v_{11} + +w_{11}$	$9 + +2 \cdot v_{11} + +w_{11}$	$16 + +2 \cdot v_{11} + +2 \cdot w_{11}$	$23 + +3 \cdot v_{11} + +2 \cdot w_{11}$
Earliest possible service times for transfer node 2	00	$7 + v_{11}$	$14 + +v_{11} + +w_{11}$	$21 + +2 \cdot v_{11} + +w_{11}$	----	----
Direction 2, order of served transfer nodes 2 → 1						
Earliest possible service times for transfer node 2	-12	$-5 + +v_{11}$	$2 + +v_{11} + +w_{11}$	$9 + +2 \cdot v_{11} + +w_{11}$	$16 + +2 \cdot v_{11} + +2 \cdot w_{11}$	$23 + +3 \cdot v_{11} + +2 \cdot w_{11}$
Earliest possible service times for transfer node 1	00	$7 + v_{11}$	$14 + +v_{11} + +w_{11}$	$21 + +2 \cdot v_{11} + +w_{11}$	----	----

Table 7.12: Calculation of the earliest possible service times of transfer nodes 1 and 2

The earliest possible time positions of the services of transfer node 1 by the outgoing line 2 in both directions occur at 00, 10, 20, 30 and of transfer node 2 by the incoming line 3 at 00, 10, 20.

*In the next procedure, we identify the values of the incidence matrix **B**. For the elements of the matrix, based on the information given in Table 7.2 (independent of the number of connections)* $b_{11121} = 1$, $b_{11122} = 1$ *and* $b_{23212} = 1$, *for other combinations of indices* $b_{uiljs} = 0$ *is valid.*

In the next procedure we determine the values of the maximum allowed time shifts (delays) of the coordinated lines. Since only the connections of line 1 in directions 1 and 2, the connections of line 2 in directions 1 and 2 and the connections of line 3 in direction 2 are involved in the network node time coordination, we will allow time shifts and therefore maximum time shifts only for those lines and directions. Therefore, assuming that all time positions can be used within the headway on the lines, the following $a_{11} = 6$, $a_{12} = 6$, $a_{21} = 9$, $a_{22} = 9$ *and* $a_{32} = 9$ *will apply .*

The input variables are then substituted into a mathematical model of the form (only the terms of the objective function and the constraints for the required coordination links will be included in the model for illustrative purposes, i.e., when $b_{uiljs} = 1$, *the obligatory constraints will be partially written out in abbreviated form):*

$$min\, f(x,h,z,v,w) = 1 \cdot h_{111121} + 1 \cdot h_{111221} + 1 \cdot h_{111321} + 1 \cdot h_{111421} +$$
$$+1 \cdot h_{111122} + 1 \cdot h_{111222} + 1 \cdot h_{111322} + 1 \cdot h_{111422} +$$
$$+1 \cdot h_{232112} + 1 \cdot h_{232212} + 1 \cdot h_{232312}$$

subject to:

constraints for a coordination link characterised by an element $b_{11121} = 1$*:*

$$[0 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111211} - 1)$$
$$[0 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111211})$$
$$[10 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111212} - 1)$$
$$[10 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111212})$$
$$[20 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111213} - 1)$$
$$[20 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111213})$$

$$[30 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111214} - 1)$$
$$[30 + x_{21}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111121} + M \cdot (1 - z_{1111214})$$
$$z_{1111211} + z_{1111212} + z_{1111213} + z_{1111214} = 1$$
$$[0 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112211} - 1)$$
$$[0 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111221} + M \cdot (1 - z_{1112211})$$
$$[10 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112212} - 1)$$
$$[10 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111221} + M \cdot (1 - z_{1112212})$$
$$[20 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112213} - 1)$$
$$[20 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111221} + M \cdot (1 - z_{1112213})$$
$$[30 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112214} - 1)$$
$$[30 + x_{21}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111221} + M \cdot (1 - z_{1112214})$$
$$z_{1112211} + z_{1112212} + z_{1112213} + z_{1112214} = 1$$
$$[0 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113211} - 1)$$
$$[0 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111321} + M \cdot (1 - z_{1113211})$$
$$[10 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113212} - 1)$$
$$[10 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111321} + M \cdot (1 - z_{1113212})$$
$$[20 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113213} - 1)$$
$$[20 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111321} + M \cdot (1 - z_{1113213})$$
$$[30 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113214} - 1)$$
$$[30 + x_{21}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111321} + M \cdot (1 - z_{1113214})$$
$$z_{1113211} + z_{1113212} + z_{1113213} + z_{1113214} = 1$$
$$[0 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114211} - 1)$$
$$[0 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111421} + M \cdot (1 - z_{1114211})$$
$$[10 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114212} - 1)$$
$$[10 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111421} + M \cdot (1 - z_{1114212})$$
$$[20 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114213} - 1)$$
$$[20 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111421} + M \cdot (1 - z_{1114213})$$
$$[30 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114214} - 1)$$
$$[30 + x_{21}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111421} + M \cdot (1 - z_{1114214})$$
$$z_{1114211} + z_{1114212} + z_{1114213} + z_{1114214} = 1$$

constraints for a coordination link characterised by an element $b_{11122} = 1$*:*

$$[0 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111221} - 1)$$

$$[0 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111221})$$
$$[10 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111222} - 1)$$
$$[10 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111222})$$
$$[20 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111223} - 1)$$
$$[20 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111223})$$
$$[30 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1111224} - 1)$$
$$[30 + x_{22}] - [2 + v_{11} + w_{11} + x_{11} + 5] \leq h_{111122} + M \cdot (1 - z_{1111224})$$
$$z_{1111221} + z_{1111222} + z_{1111223} + z_{1111224} = 1$$
$$[0 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112221} - 1)$$
$$[0 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111222} + M \cdot (1 - z_{1112221})$$
$$[10 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112222} - 1)$$
$$[10 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111222} + M \cdot (1 - z_{1112222})$$
$$[20 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112223} - 1)$$
$$[20 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111222} + M \cdot (1 - z_{1112223})$$
$$[30 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \geq M \cdot (z_{1112224} - 1)$$
$$[30 + x_{22}] - [9 + 2 \cdot v_{11} + w_{11} + x_{11} + 5] \leq h_{111222} + M \cdot (1 - z_{1112224})$$
$$z_{1112221} + z_{1112222} + z_{1112223} + z_{1112224} = 1$$
$$[0 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113221} - 1)$$
$$[0 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111322} + M \cdot (1 - z_{1113221})$$
$$[10 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113222} - 1)$$
$$[10 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111322} + M \cdot (1 - z_{1113222})$$
$$[20 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113223} - 1)$$
$$[20 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111322} + M \cdot (1 - z_{1113223})$$
$$[30 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1113224} - 1)$$
$$[30 + x_{22}] - [16 + 2 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111322} + M \cdot (1 - z_{1113224})$$
$$z_{1113221} + z_{1113222} + z_{1113223} + z_{1113224} = 1$$
$$[0 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114221} - 1)$$
$$[0 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111422} + M \cdot (1 - z_{1114221})$$
$$[10 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114222} - 1)$$
$$[10 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111422} + M \cdot (1 - z_{1114222})$$
$$[20 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114223} - 1)$$
$$[20 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111422} + M \cdot (1 - z_{1114223})$$
$$[30 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \geq M \cdot (z_{1114224} - 1)$$
$$[30 + x_{22}] - [23 + 3 \cdot v_{11} + 2 \cdot w_{11} + x_{11} + 5] \leq h_{111422} + M \cdot (1 - z_{1114224})$$
$$z_{1114221} + z_{1114222} + z_{1114223} + z_{1114224} = 1$$
$$v_{11} + w_{11} = 1$$

constraints for a coordination link characterised by an element $b_{23212} = 1$*:*

$$[2 + v_{12} + w_{12} + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321121} - 1)$$

$$[2 + v_{12} + w_{12} + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321121})$$

$$[9 + 2 \cdot v_{12} + w_{12} + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321122} - 1)$$

$$[9 + 2 \cdot v_{12} + w_{12} + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321122})$$
$$[16 + 2 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321123} - 1)$$
$$[16 + 2 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321123})$$
$$[23 + 3 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321124} - 1)$$
$$[23 + 3 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321124})$$
$$[30 + 3 \cdot v_{12} + 3 \cdot w_{12} + x_{12}] - [0 + x_{32} + 6] \geq M \cdot (z_{2321125} - 1)$$
$$[30 + 3 \cdot v_{12} + 3 \cdot w_{12} + x_{12}] - [0 + x_{32} + 6] \leq h_{232112} + M \cdot (1 - z_{2321125})$$
$$z_{2321121} + z_{2321122} + z_{2321123} + z_{2321124} + z_{2321125} = 1$$
$$[2 + v_{12} + w_{12} + x_{12}] - [10 + x_{32} + 6] \geq M \cdot (z_{2322121} - 1)$$
$$[2 + v_{12} + w_{12} + x_{12}] - [10 + x_{32} + 6] \leq h_{232212} + M \cdot (1 - z_{2322121})$$
$$[9 + 2 \cdot v_{12} + w_{12} + x_{12}] - [10 + x_{32} + 6] \geq M \cdot (z_{2322122} - 1)$$
$$[9 + 2 \cdot v_{12} + w_{12} + x_{12}] - [10 + x_{32} + 6] \leq h_{232212} + M \cdot (1 - z_{2322122})$$
$$[16 + 2 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [10 + x_{32} + 6] \geq M \cdot (z_{2322123} - 1)$$
$$[16 + 2 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [10 + x_{32} + 6] \leq h_{232212} + M \cdot (1 - z_{2322123})$$
$$[23 + 3 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [10 + x_{32} + 6] \geq M \cdot (z_{2322124} - 1)$$
$$[23 + 3 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [10 + x_{32} + 6] \leq h_{232212} + M \cdot (1 - z_{2322124})$$
$$[30 + 3 \cdot v_{12} + 3 \cdot w_{12} + x_{12}] - [10 + x_{32} + 6] \geq M \cdot (z_{2322125} - 1)$$
$$[30 + 3 \cdot v_{12} + 3 \cdot w_{12} + x_{12}] - [10 + x_{32} + 6] \leq h_{232212} + M \cdot (1 - z_{2322125})$$
$$z_{2322121} + z_{2322122} + z_{2322123} + z_{2322124} + z_{2322125} = 1$$
$$[2 + v_{12} + w_{12} + x_{12}] - [20 + x_{32} + 6] \geq M \cdot (z_{2323121} - 1)$$
$$[2 + v_{12} + w_{12} + x_{12}] - [20 + x_{32} + 6] \leq h_{232312} + M \cdot (1 - z_{2323121})$$
$$[9 + 2 \cdot v_{12} + w_{12} + x_{12}] - [20 + x_{32} + 6] \geq M \cdot (z_{2323122} - 1)$$
$$[9 + 2 \cdot v_{12} + w_{12} + x_{12}] - [20 + x_{32} + 6] \leq h_{232312} + M \cdot (1 - z_{2323122})$$
$$[16 + 2 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [20 + x_{32} + 6] \geq M \cdot (z_{2323123} - 1)$$
$$[16 + 2 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [20 + x_{32} + 6] \leq h_{232312} + M \cdot (1 - z_{2323123})$$
$$[23 + 3 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [20 + x_{32} + 6] \geq M \cdot (z_{2323124} - 1)$$
$$[23 + 3 \cdot v_{12} + 2 \cdot w_{12} + x_{12}] - [20 + x_{32} + 6] \leq h_{232312} + M \cdot (1 - z_{2323124})$$
$$[30 + 3 \cdot v_{12} + 3 \cdot w_{12} + x_{12}] - [20 + x_{32} + 6] \geq M \cdot (z_{2323125} - 1)$$
$$[30 + 3 \cdot v_{12} + 3 \cdot w_{12} + x_{12}] - [20 + x_{32} + 6] \leq h_{232312} + M \cdot (1 - z_{2323125})$$
$$z_{2323121} + z_{2323122} + z_{2323123} + z_{2323124} + z_{2323125} = 1$$

constraints ensuring that the maximum permitted time shifts of connections on coordinated lines are not exceeded:

$$v_{12} + w_{12} = 1$$

$$x_{11} \leq 6$$
$$x_{12} \leq 6$$
$$x_{21} \leq 9$$
$$x_{22} \leq 9$$
$$x_{31} \leq 0$$
$$x_{32} \leq 9$$

obligatory constraints:

$$x_{il} \in R_0^+ \qquad \text{for } i \in \{1;2;3\} \text{ and } l \in \{1;2\}$$

$h_{111121} \in R_0^+$	$h_{111221} \in R_0^+$	$h_{111321} \in R_0^+$	$h_{111421} \in R_0^+$
$h_{111122} \in R_0^+$	$h_{111222} \in R_0^+$	$h_{111322} \in R_0^+$	$h_{111422} \in R_0^+$
$h_{232112} \in R_0^+$	$h_{232212} \in R_0^+$	$h_{232312} \in R_0^+$	
$z_{1111211} \in \{0;1\}$	$z_{1111212} \in \{0;1\}$	$z_{1111213} \in \{0;1\}$	$z_{1111214} \in \{0;1\}$
$z_{1112211} \in \{0;1\}$	$z_{1112212} \in \{0;1\}$	$z_{1112213} \in \{0;1\}$	$z_{1112214} \in \{0;1\}$
$z_{1113211} \in \{0;1\}$	$z_{1113212} \in \{0;1\}$	$z_{1113213} \in \{0;1\}$	$z_{1113214} \in \{0;1\}$
$z_{1114211} \in \{0;1\}$	$z_{1114212} \in \{0;1\}$	$z_{1114213} \in \{0;1\}$	$z_{1114214} \in \{0;1\}$
$z_{1111221} \in \{0;1\}$	$z_{1111222} \in \{0;1\}$	$z_{1111223} \in \{0;1\}$	$z_{1111224} \in \{0;1\}$
$z_{1112221} \in \{0;1\}$	$z_{1112222} \in \{0;1\}$	$z_{1112223} \in \{0;1\}$	$z_{1112224} \in \{0;1\}$
$z_{1113221} \in \{0;1\}$	$z_{1113222} \in \{0;1\}$	$z_{1113223} \in \{0;1\}$	$z_{1113224} \in \{0;1\}$
$z_{1114221} \in \{0;1\}$	$z_{1114222} \in \{0;1\}$	$z_{1114223} \in \{0;1\}$	$z_{1114224} \in \{0;1\}$
$z_{2321121} \in \{0;1\}$	$z_{2321122} \in \{0;1\}$	$z_{2321123} \in \{0;1\}$	$z_{2321124} \in \{0;1\}$
$z_{2321125} \in \{0;1\}$	$z_{2322121} \in \{0;1\}$	$z_{2322122} \in \{0;1\}$	$z_{2322123} \in \{0;1\}$
$z_{2322124} \in \{0;1\}$	$z_{2322125} \in \{0;1\}$	$z_{2323121} \in \{0;1\}$	$z_{2323122} \in \{0;1\}$
$z_{2323123} \in \{0;1\}$	$z_{2323124} \in \{0;1\}$	$z_{2323125} \in \{0;1\}$	

The model text that can be solved in the Xpress-IVE optimization software is as follows:

```
model Example_7_3
uses "mmxprs";
declarations
```

```
node=1..2
line=1..3
direction=1..2
connection=1..5
t:array(node,line,direction,connection)of real
a:array(line,direction)of real
x:array(line,direction)of mpvar
z:array(node,line,direction,connection,line,direction,connection)of mpvar
h:array(node,line,direction,connection,line,direction)of mpvar
f:array(node,line,direction,connection,line,direction)of real
trest:array(node,line,line)of real
v:array(1..1,direction)of mpvar
w:array(1..1,direction)of mpvar
end-declarations
M:=1000000
(0+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,1,1)-1)
(0+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,1))
(10+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,1,2)-1)
(10+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,2))
(20+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,1,2)-1)
(20+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,3))
(30+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,1,4)-1)
(30+x(2,1))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,1)+M*(1-z(1,1,1,1,2,1,4))
z(1,1,1,1,2,1,1)+z(1,1,1,1,2,1,2)+z(1,1,1,1,2,1,3)+z(1,1,1,1,2,1,4)=1
(0+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,1,1)-1)
(0+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,1)+M*(1-z(1,1,1,2,2,1,1))
(10+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,1,2)-1)
(10+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,1)+M*(1-z(1,1,1,2,2,1,2))
(20+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,1,3)-1)
(20+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,1)+M*(1-z(1,1,1,2,2,1,3))
(30+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,1,4)-1)
(30+x(2,1))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,1)+M*(1-z(1,1,1,2,2,1,4))
z(1,1,1,2,2,1,1)+z(1,1,1,2,2,1,2)+z(1,1,1,2,2,1,3)+z(1,1,1,2,2,1,4)=1
(0+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,1,1)-1)
(0+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,1)+M*(1-z(1,1,1,3,2,1,1))
(10+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,1,2)-1)
(10+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,1)+M*(1-z(1,1,1,3,2,1,2))
(20+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,1,3)-1)
(20+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,1)+M*(1-z(1,1,1,3,2,1,3))
(30+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,1,4)-1)
(30+x(2,1))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,1)+M*(1-z(1,1,1,3,2,1,4))
z(1,1,1,3,2,1,1)+z(1,1,1,3,2,1,2)+z(1,1,1,3,2,1,3)+z(1,1,1,3,2,1,4)=1
(0+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,1,1)-1)
(0+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,1)+M*(1-z(1,1,1,4,2,1,1))
(10+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,1,2)-1)
(10+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,1)+M*(1-z(1,1,1,4,2,1,2))
(20+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,1,3)-1)
(20+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,1)+M*(1-z(1,1,1,4,2,1,3))
```

(30+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,1,4)-1)
(30+x(2,1))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,1)+M*(1-z(1,1,1,4,2,1,4))
z(1,1,1,4,2,1,1)+z(1,1,1,4,2,1,2)+z(1,1,1,4,2,1,3)+z(1,1,1,4,2,1,4)=1
(0+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,2,1)-1)
(0+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,1))
(10+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,2,2)-1)
(10+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,2))
(20+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,2,2)-1)
(20+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,3))
(30+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,1,2,2,4)-1)
(30+x(2,2))-(2+v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,1,2,2)+M*(1-z(1,1,1,1,2,2,4))
z(1,1,1,1,2,2,1)+z(1,1,1,1,2,2,2)+z(1,1,1,1,2,2,3)+z(1,1,1,1,2,2,4)=1
(0+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,2,1)-1)
(0+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,2)+M*(1-z(1,1,1,2,2,2,1))
(10+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,2,2)-1)
(10+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,2)+M*(1-z(1,1,1,2,2,2,2))
(20+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,2,3)-1)
(20+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,2)+M*(1-z(1,1,1,2,2,2,3))
(30+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)>=M*(z(1,1,1,2,2,2,4)-1)
(30+x(2,2))-(9+2*v(1,1)+w(1,1)+x(1,1)+5)<=h(1,1,1,2,2,2)+M*(1-z(1,1,1,2,2,2,4))
z(1,1,1,2,2,2,1)+z(1,1,1,2,2,2,2)+z(1,1,1,2,2,2,3)+z(1,1,1,2,2,2,4)=1
(0+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,2,1)-1)
(0+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,2)+M*(1-z(1,1,1,3,2,2,1))
(10+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,2,2)-1)
(10+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,2)+M*(1-z(1,1,1,3,2,2,2))
(20+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,2,3)-1)
(20+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,2)+M*(1-z(1,1,1,3,2,2,3))
(30+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,3,2,2,4)-1)
(30+x(2,2))-(16+2*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,3,2,2)+M*(1-z(1,1,1,3,2,2,4))
z(1,1,1,3,2,2,1)+z(1,1,1,3,2,2,2)+z(1,1,1,3,2,2,3)+z(1,1,1,3,2,2,4)=1
(0+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,2,1)-1)
(0+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,2)+M*(1-z(1,1,1,4,2,2,1))
(10+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,2,2)-1)
(10+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,2)+M*(1-z(1,1,1,4,2,2,2))
(20+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,2,3)-1)
(20+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,2)+M*(1-z(1,1,1,4,2,2,3))
(30+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)>=M*(z(1,1,1,4,2,2,4)-1)
(30+x(2,2))-(23+3*v(1,1)+2*w(1,1)+x(1,1)+5)<=h(1,1,1,4,2,2)+M*(1-z(1,1,1,4,2,2,4))
z(1,1,1,4,2,2,1)+z(1,1,1,4,2,2,2)+z(1,1,1,4,2,2,3)+z(1,1,1,4,2,2,4)=1
v(1,1)+w(1,1)=1

(2+v(1,2)+w(1,2)+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,1)-1)
(2+v(1,2)+w(1,2)+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,1))
(9+2*v(1,2)+w(1,2)+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,2)-1)
(9+2*v(1,2)+w(1,2)+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,2))
(16+2*v(1,2)+2*w(1,2)+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,3)-1)
(16+2*v(1,2)+2*w(1,2)+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,3))
(23+3*v(1,2)+2*w(1,2)+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,4)-1)

```
(23+3*v(1,2)+2*w(1,2)+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,4))
(30+3*v(1,2)+3*w(1,2)+x(1,2))-(0+x(3,2)+6)>=M*(z(2,3,2,1,1,2,5)-1)
(30+3*v(1,2)+3*w(1,2)+x(1,2))-(0+x(3,2)+6)<=h(2,3,2,1,1,2)+M*(1-z(2,3,2,1,1,2,5))
z(2,3,2,1,1,2,1)+z(2,3,2,1,1,2,2)+z(2,3,2,1,1,2,3)+z(2,3,2,1,1,2,4)+z(2,3,2,1,1,2,5)=1

(2+v(1,2)+w(1,2)+x(1,2))-(10+x(3,2)+6)>=M*(z(2,3,2,2,1,2,1)-1)
(2+v(1,2)+w(1,2)+x(1,2))-(10+x(3,2)+6)<=h(2,3,2,2,1,2)+M*(1-z(2,3,2,2,1,2,1))
(9+2*v(1,2)+w(1,2)+x(1,2))-(10+x(3,2)+6)>=M*(z(2,3,2,2,1,2,2)-1)
(9+2*v(1,2)+w(1,2)+x(1,2))-(10+x(3,2)+6)<=h(2,3,2,2,2,2)+M*(1-z(2,3,2,2,1,2,2))
(16+2*v(1,2)+2*w(1,2)+x(1,2))-(10+x(3,2)+6)>=M*(z(2,3,2,2,1,2,3)-1)
(16+2*v(1,2)+2*w(1,2)+x(1,2))-(10+x(3,2)+6)<=h(2,3,2,2,1,2)+M*(1-z(2,3,2,2,1,2,3))
(23+3*v(1,2)+2*w(1,2)+x(1,2))-(10+x(3,2)+6)>=M*(z(2,3,2,2,1,2,4)-1)
(23+3*v(1,2)+2*w(1,2)+x(1,2))-(10+x(3,2)+6)<=h(2,3,2,2,1,2)+M*(1-z(2,3,2,2,1,2,4))
(30+3*v(1,2)+3*w(1,2)+x(1,2))-(10+x(3,2)+6)>=M*(z(2,3,2,2,1,2,5)-1)
(30+3*v(1,2)+3*w(1,2)+x(1,2))-(10+x(3,2)+6)<=h(2,3,2,2,1,2)+M*(1-z(2,3,2,2,1,2,5))
z(2,3,2,2,1,2,1)+z(2,3,2,2,1,2,2)+z(2,3,2,2,1,2,3)+z(2,3,2,2,1,2,4)+z(2,3,2,2,1,2,5)=1

(2+v(1,2)+w(1,2)+x(1,2))-(20+x(3,2)+6)>=M*(z(2,3,2,3,1,2,1)-1)
(2+v(1,2)+w(1,2)+x(1,2))-(20+x(3,2)+6)<=h(2,3,2,3,1,2)+M*(1-z(2,3,2,3,1,2,1))
(9+2*v(1,2)+w(1,2)+x(1,2))-(20+x(3,2)+6)>=M*(z(2,3,2,3,1,2,2)-1)
(9+2*v(1,2)+w(1,2)+x(1,2))-(20+x(3,2)+6)<=h(2,3,2,3,2,2)+M*(1-z(2,3,2,3,1,2,2))
(16+2*v(1,2)+2*w(1,2)+x(1,2))-(20+x(3,2)+6)>=M*(z(2,3,2,3,1,2,3)-1)
(16+2*v(1,2)+2*w(1,2)+x(1,2))-(20+x(3,2)+6)<=h(2,3,2,3,1,2)+M*(1-z(2,3,2,3,1,2,3))
(23+3*v(1,2)+2*w(1,2)+x(1,2))-(20+x(3,2)+6)>=M*(z(2,3,2,3,1,2,4)-1)
(23+3*v(1,2)+2*w(1,2)+x(1,2))-(20+x(3,2)+6)<=h(2,3,2,3,1,2)+M*(1-z(2,3,2,3,1,2,4))
(30+3*v(1,2)+3*w(1,2)+x(1,2))-(20+x(3,2)+6)>=M*(z(2,3,2,3,1,2,5)-1)
(30+3*v(1,2)+3*w(1,2)+x(1,2))-(20+x(3,2)+6)<=h(2,3,2,3,1,2)+M*(1-z(2,3,2,3,1,2,5))
z(2,3,2,3,1,2,1)+z(2,3,2,3,1,2,2)+z(2,3,2,3,1,2,3)+z(2,3,2,3,1,2,4)+z(2,3,2,3,1,2,5)=1

v(1,2)+w(1,2)=1
x(1,1)<=6
x(1,2)<=6
x(2,1)<=9
x(2,2)<=9
x(3,2)<=9
v(1,1)is_binary
v(1,2)is_binary
w(1,1)is_binary
w(1,2)is_binary
z(1,1,1,1,2,1,1)is_binary
z(1,1,1,1,2,1,2)is_binary
z(1,1,1,1,2,1,3)is_binary
z(1,1,1,1,2,1,4)is_binary
z(1,1,1,2,2,1,1)is_binary
z(1,1,1,2,2,1,2)is_binary
z(1,1,1,2,2,1,3)is_binary
z(1,1,1,2,2,1,4)is_binary
z(1,1,1,3,2,1,1)is_binary
```

```
z(1,1,1,3,2,1,2)is_binary
z(1,1,1,3,2,1,3)is_binary
z(1,1,1,3,2,1,4)is_binary
z(1,1,1,4,2,1,1)is_binary
z(1,1,1,4,2,1,2)is_binary
z(1,1,1,4,2,1,3)is_binary
z(1,1,1,4,2,1,4)is_binary
z(1,1,1,1,2,2,1)is_binary
z(1,1,1,1,2,2,2)is_binary
z(1,1,1,1,2,2,3)is_binary
z(1,1,1,1,2,2,4)is_binary
z(1,1,1,2,2,2,1)is_binary
z(1,1,1,2,2,2,2)is_binary
z(1,1,1,2,2,2,3)is_binary
z(1,1,1,2,2,2,4)is_binary
z(1,1,1,3,2,2,1)is_binary
z(1,1,1,3,2,2,2)is_binary
z(1,1,1,3,2,2,3)is_binary
z(1,1,1,3,2,2,4)is_binary
z(1,1,1,4,2,2,1)is_binary
z(1,1,1,4,2,2,2)is_binary
z(1,1,1,4,2,2,3)is_binary
z(1,1,1,4,2,2,4)is_binary
z(2,3,2,1,1,2,1)is_binary
z(2,3,2,1,1,2,2)is_binary
z(2,3,2,1,1,2,3)is_binary
z(2,3,2,1,1,2,4)is_binary
z(2,3,2,1,1,2,5)is_binary
z(2,3,2,2,1,2,1)is_binary
z(2,3,2,2,1,2,2)is_binary
z(2,3,2,2,1,2,3)is_binary
z(2,3,2,2,1,2,4)is_binary
z(2,3,2,2,1,2,5)is_binary
z(2,3,2,3,1,2,1)is_binary
z(2,3,2,3,1,2,2)is_binary
z(2,3,2,3,1,2,3)is_binary
z(2,3,2,3,1,2,4)is_binary
z(2,3,2,3,1,2,5)is_binary
Total_time_loss:=1*h(1,1,1,1,2,1)+1*h(1,1,1,2,2,1)+1*h(1,1,1,3,2,1)+1*h(1,1,1,4,2,1)+1*h(1,1,1,1,2,2)
+1*h(1,1,1,2,2,2)+1*h(1,1,1,3,2,2)+1*h(1,1,1,4,2,2)+1*h(2,3,2,1,1,2)+1*h(2,3,2,2,1,2)+1*h(2,3,2,3,1,2
)
minimize(Total_time_loss)
writeln("Total time loss is: ",getobjval," person-minutes")
writeln("Time shifts of individual lines:")
writeln("x(1,1) = ",getsol(x(1,1))," min")
writeln("x(1,2) = ",getsol(x(1,2))," min")
writeln("x(2,1) = ",getsol(x(2,1))," min")
writeln("x(2,2) = ",getsol(x(2,2))," min")
```

```
writeln("x(3,1) = ",getsol(x(3,1))," min")
writeln("x(3,2) = ",getsol(x(3,2))," min")
writeln("Waiting:")
writeln("h(1,1,1,1,2,1) = ",getsol(h(1,1,1,1,2,1))," min")
writeln("h(1,1,1,2,2,1) = ",getsol(h(1,1,1,2,2,1))," min")
writeln("h(1,1,1,3,2,1) = ",getsol(h(1,1,1,3,2,1))," min")
writeln("h(1,1,1,4,2,1) = ",getsol(h(1,1,1,4,2,1))," min")
writeln("h(1,1,1,1,2,2) = ",getsol(h(1,1,1,1,2,2))," min")
writeln("h(1,1,1,2,2,2) = ",getsol(h(1,1,1,2,2,2))," min")
writeln("h(1,1,1,3,2,2) = ",getsol(h(1,1,1,3,2,2))," min")
writeln("h(1,1,1,4,2,2) = ",getsol(h(1,1,1,4,2,2))," min")
writeln("h(2,3,2,1,1,2) = ",getsol(h(2,3,2,1,1,2))," min")
writeln("h(2,3,2,2,1,2) = ",getsol(h(2,3,2,2,1,2))," min")
writeln("h(2,3,2,3,1,2) = ",getsol(h(2,3,2,3,1,2))," min")
writeln("Coordination links:")
writeln("z(1,1,1,1,2,1,1) = ",getsol(z(1,1,1,1,2,1,1)))
writeln("z(1,1,1,1,2,1,2) = ",getsol(z(1,1,1,1,2,1,2)))
writeln("z(1,1,1,1,2,1,3) = ",getsol(z(1,1,1,1,2,1,3)))
writeln("z(1,1,1,1,2,1,4) = ",getsol(z(1,1,1,1,2,1,4)))
writeln("z(1,1,1,2,2,1,1) = ",getsol(z(1,1,1,2,2,1,1)))
writeln("z(1,1,1,2,2,1,2) = ",getsol(z(1,1,1,2,2,1,2)))
writeln("z(1,1,1,2,2,1,3) = ",getsol(z(1,1,1,2,2,1,3)))
writeln("z(1,1,1,2,2,1,4) = ",getsol(z(1,1,1,2,2,1,4)))
writeln("z(1,1,1,3,2,1,1) = ",getsol(z(1,1,1,3,2,1,1)))
writeln("z(1,1,1,3,2,1,2) = ",getsol(z(1,1,1,3,2,1,2)))
writeln("z(1,1,1,3,2,1,3) = ",getsol(z(1,1,1,3,2,1,3)))
writeln("z(1,1,1,3,2,1,4) = ",getsol(z(1,1,1,3,2,1,4)))
writeln("z(1,1,1,4,2,1,1) = ",getsol(z(1,1,1,4,2,1,1)))
writeln("z(1,1,1,4,2,1,2) = ",getsol(z(1,1,1,4,2,1,2)))
writeln("z(1,1,1,4,2,1,3) = ",getsol(z(1,1,1,4,2,1,3)))
writeln("z(1,1,1,4,2,1,4) = ",getsol(z(1,1,1,4,2,1,4)))
writeln("z(1,1,1,1,2,2,1) = ",getsol(z(1,1,1,1,2,2,1)))
writeln("z(1,1,1,1,2,2,2) = ",getsol(z(1,1,1,1,2,2,2)))
writeln("z(1,1,1,1,2,2,3) = ",getsol(z(1,1,1,1,2,2,3)))
writeln("z(1,1,1,1,2,2,4) = ",getsol(z(1,1,1,1,2,2,4)))
writeln("z(1,1,1,2,2,2,1) = ",getsol(z(1,1,1,2,2,2,1)))
writeln("z(1,1,1,2,2,2,2) = ",getsol(z(1,1,1,2,2,2,2)))
writeln("z(1,1,1,2,2,2,3) = ",getsol(z(1,1,1,2,2,2,3)))
writeln("z(1,1,1,2,2,2,4) = ",getsol(z(1,1,1,2,2,2,4)))
writeln("z(1,1,1,3,2,2,1) = ",getsol(z(1,1,1,3,2,2,1)))
writeln("z(1,1,1,3,2,2,2) = ",getsol(z(1,1,1,3,2,2,2)))
writeln("z(1,1,1,3,2,2,3) = ",getsol(z(1,1,1,3,2,2,3)))
writeln("z(1,1,1,3,2,2,4) = ",getsol(z(1,1,1,3,2,2,4)))
writeln("z(1,1,1,4,2,2,1) = ",getsol(z(1,1,1,4,2,2,1)))
writeln("z(1,1,1,4,2,2,2) = ",getsol(z(1,1,1,4,2,2,2)))
writeln("z(1,1,1,4,2,2,3) = ",getsol(z(1,1,1,4,2,2,3)))
writeln("z(1,1,1,4,2,2,4) = ",getsol(z(1,1,1,4,2,2,4)))
writeln("z(2,3,2,1,1,2,1) = ",getsol(z(2,3,2,1,1,2,1)))
```

```
writeln("z(2,3,2,1,1,2,2) = ",getsol(z(2,3,2,1,1,2,2)))
writeln("z(2,3,2,1,1,2,3) = ",getsol(z(2,3,2,1,1,2,3)))
writeln("z(2,3,2,1,1,2,4) = ",getsol(z(2,3,2,1,1,2,4)))
writeln("z(2,3,2,1,1,2,5) = ",getsol(z(2,3,2,1,1,2,5)))
writeln("z(2,3,2,2,1,2,1) = ",getsol(z(2,3,2,2,1,2,1)))
writeln("z(2,3,2,2,1,2,2) = ",getsol(z(2,3,2,2,1,2,2)))
writeln("z(2,3,2,2,1,2,3) = ",getsol(z(2,3,2,2,1,2,3)))
writeln("z(2,3,2,2,1,2,4) = ",getsol(z(2,3,2,2,1,2,4)))
writeln("z(2,3,2,2,1,2,5) = ",getsol(z(2,3,2,2,1,2,5)))
writeln("z(2,3,2,3,1,2,1) = ",getsol(z(2,3,2,3,1,2,1)))
writeln("z(2,3,2,3,1,2,2) = ",getsol(z(2,3,2,3,1,2,2)))
writeln("z(2,3,2,3,1,2,3) = ",getsol(z(2,3,2,3,1,2,3)))
writeln("z(2,3,2,3,1,2,4) = ",getsol(z(2,3,2,3,1,2,4)))
writeln("z(2,3,2,3,1,2,5) = ",getsol(z(2,3,2,3,1,2,5)))
writeln("Variables modelling alternating headway:")
writeln("v(1,1) = ",getsol(v(1,1)))
writeln("w(1,1) = ",getsol(w(1,1)))
writeln("v(1,2) = ",getsol(v(1,2)))
writeln("w(1,2) = ",getsol(w(1,2)))
end-model
```

After the optimization calculation was completed, the following results were obtained:

Total time loss is: 35 person-minutes
Time shifts of individual lines:
x(1,1) = 0 min
x(1,2) = 2 min
x(2,1) = 3 min
x(2,2) = 3 min
x(3,1) = 0 min
x(3,2) = 9 min
Waiting:
h(1,1,1,1,2,1) = 5 min
h(1,1,1,2,2,1) = 7 min
h(1,1,1,3,2,1) = 0 min
h(1,1,1,4,2,1) = 2 min
h(1,1,1,1,2,2) = 5 min
h(1,1,1,2,2,2) = 7 min
h(1,1,1,3,2,2) = 0 min
h(1,1,1,4,2,2) = 2 min
h(2,3,2,1,1,2) = 5 min
h(2,3,2,2,1,2) = 2 min
h(2,3,2,3,1,2) = 0 min
Coordination links:
z(1,1,1,1,2,1,1) = 0
z(1,1,1,1,2,1,2) = 1
z(1,1,1,1,2,1,3) = 0

z(1,1,1,1,2,1,4) = 0
z(1,1,1,2,2,1,1) = 0
z(1,1,1,2,2,1,2) = 0
z(1,1,1,2,2,1,3) = 1
z(1,1,1,2,2,1,4) = 0
z(1,1,1,3,2,1,1) = 0
z(1,1,1,3,2,1,2) = 0
z(1,1,1,3,2,1,3) = 1
z(1,1,1,3,2,1,4) = 0
z(1,1,1,4,2,1,1) = 0
z(1,1,1,4,2,1,2) = 0
z(1,1,1,4,2,1,3) = 0
z(1,1,1,4,2,1,4) = 1
z(1,1,1,1,2,2,1) = 0
z(1,1,1,1,2,2,2) = 1
z(1,1,1,1,2,2,3) = 0
z(1,1,1,1,2,2,4) = 0
z(1,1,1,2,2,2,1) = 0
z(1,1,1,2,2,2,2) = 0
z(1,1,1,2,2,2,3) = 1
z(1,1,1,2,2,2,4) = 0
z(1,1,1,3,2,2,1) = 0
z(1,1,1,3,2,2,2) = 0
z(1,1,1,3,2,2,3) = 1
z(1,1,1,3,2,2,4) = 0
z(1,1,1,4,2,2,1) = 0
z(1,1,1,4,2,2,2) = 0
z(1,1,1,4,2,2,3) = 0
z(1,1,1,4,2,2,4) = 1
z(2,3,2,1,1,2,1) = 0
z(2,3,2,1,1,2,2) = 0
z(2,3,2,1,1,2,3) = 1
z(2,3,2,1,1,2,4) = 0
z(2,3,2,1,1,2,5) = 0
z(2,3,2,2,1,2,1) = 0
z(2,3,2,2,1,2,2) = 0
z(2,3,2,2,1,2,3) = 0
z(2,3,2,2,1,2,4) = 1
z(2,3,2,2,1,2,5) = 0
z(2,3,2,3,1,2,1) = 0
z(2,3,2,3,1,2,2) = 0
z(2,3,2,3,1,2,3) = 0
z(2,3,2,3,1,2,4) = 0
z(2,3,2,3,1,2,5) = 1
Variables modelling alternating headway:
v(1,1) = 1
w(1,1) = 0
v(1,2) = 0

w(1,2) = 1

The results can be interpreted as follows:

Variable values x_{il} *indicate the time shifts of the connections of the coordinated lines. In order to achieve the optimal solution (total time loss of 35 person-minutes), the following time shifts of the connections included in the network node time coordination must be performed:*

In the listing of the results of the optimization calculation $x_{11} = 0$ *minutes is stated. Variable* x_{11} *models the time shift of the connections of line 1 in direction 1 relative to their earliest possible time positions. Line 1 in direction 1 acts as an incoming line to transfer node 1, so the coordination period includes four connections serving transfer node 1 in the given direction, according to the input conditions of the task. Naturally, other connections of the line running in the same direction and serving transfer node 2 will be subject to the same time shift. The earliest possible services of transfer node 1 by line 1 connections going in direction 1 occur at 02, 09, 16 and 23 (additional connections at 30, 37, 44, 51 and 58) and of transfer node 2 at 00, 07, 14, 21 (additional connections at 28, 35, 42, 49 and 56). However, it is not only the value of the variable that is important for determining the optimal positions of service times* x_{11}*, but also the values of variables* v_{11} *and* w_{11}*, for which, according to the listing of results,* $v_{11} = 1$ *and* $w_{11} = 0$ *are valid. Variable value* v_{11} *further represents a time shift of 1 minute of the connections to which it refers. In Table 7. 13 it is possible to trace the time positions at transfer node 1 of the connections of line 1 going in direction 1 before and after the time shift.*

Number of line 1 connection running in direction 1 k	*Time position at transfer node 1 before the time shift (earliest possible time position of the connection)* $\tau_{111k}(v,w)$	*Time position at transfer node 1 after the time shift* $\tau_{111k}(v,w)+x_{11}$
1	$2+v_{11}+w_{11}$	$2+1+0+0=3$
2	$9+2\cdot v_{11}+w_{11}$	$9+2\cdot 1+0+0=11$
3	$16+2\cdot v_{11}+2\cdot w_{11}$	$16+2\cdot 1+2\cdot 0+0=18$
4	$23+3\cdot v_{11}+2\cdot w_{11}$	$23+3\cdot 1+2\cdot 0+0=26$

Table 7.13: Time positions at transfer node 1 of line 1 connections going in direction 1

As can be seen from the last column of Table 7.13, the headway between consecutive connections is indeed alternating (the values 7 and 8 alternate regularly). Other line 1 connections (connections outside the coordination period) will serve transfer node 1 in direction 2 at times 33, 41, 48 and 56.

In the results of the optimization calculation we can see that $x_{11}=2$ *minutes. Variable* x_{12} *models the time shift of line 1 connections in direction 2 relative to their earliest possible time positions. Line 1 going in direction 2 acts as the outgoing line from transfer node 2, so the coordination period includes five connections serving transfer node 2 in that direction, according to the input conditions of the problem. Naturally, other connections of the line running in the same direction and serving transfer node 1 will be subject to the same time shift. The earliest possible line 1 services of transfer node 2 running in direction 2 occur at 02, 09, 16, 23 and 30 (additional connections arrive at 37, 44, 51 and 58) and of transfer node 2 at 00, 07, 14, 21 and 28 (additional connections arrive at 35, 42, 49 and 56). However, as in the previous case, it is not only the value of the variable that is important for determining the optimal service time positions* x_{12}*, but also the values of the variables* v_{12} *and* w_{12}*, for which, according to the listing of the results,* $v_{12}=0$ *and* $w_{12}=1$ *are valid. Variable value* w_{12} *further represents a time shift of 1 minute of the connections to which it refers. In Table 7. 14* *it is possible to trace the time positions at transfer node 2, of line 1 connections going in direction 2 before and after the time shift.*

Number of line 1 connection going in direction 1 p	*Time position at transfer node 1 before the time shift (earliest possible time position of the connection)* $\tau_{212p}(v, w)$	*Time position at transfer node 1 after the time shift* $\tau_{212p}(v, w) + x_{12}$
1	$2 + v_{11} + w_{11}$	$2 + 0 + 1 + 2 = 5$
2	$9 + 2 \cdot v_{11} + w_{11}$	$9 + 2 \cdot 0 + 1 + 2 = 12$
3	$16 + 2 \cdot v_{11} + 2 \cdot w_{11}$	$16 + 2 \cdot 0 + 2 \cdot 1 + 2 = 20$
4	$23 + 3 \cdot v_{11} + 2 \cdot w_{11}$	$23 + 3 \cdot 0 + 2 \cdot 1 + 2 = 27$
5	$30 + 3 \cdot v_{11} + 3 \cdot w_{11}$	$30 + 3 \cdot 0 + 3 \cdot 1 + 2 = 35$

Table 7.14: Time positions at transfer node 2 of line 1 connections going in direction 2

As can be seen from the last column of Table 7.14, the headway between consecutive connections is indeed alternating (the values 7 and 8 alternate regularly). Other line 1 connections (connections outside the coordination period) will serve transfer node 2 in direction 2 at times 42, 50 and 57.

In the results of the optimization calculation we can see that $x_{21} = 3$ *minutes. Variable* x_{21} *models the time shift of the line 2 connections going in direction 1 relative to their earliest possible time positions. Line 2 in direction 1 acts as the outgoing line from transfer node 2, so the coordination period includes four connections serving transfer node 1 in that direction, according to the input conditions of the task. The earliest possible service of transfer node 1 by line 2 connections running in direction 1 occur at times 00, 10, 20 and 30 (additional connections arrive at times 40 and 50). This means that after applying the time shift of* $x_{21} = 3$ *minutes, the service of transfer node 1 by line 1 connections going in direction 1 will occur at times 03, 13, 23 and 33 (additional connections arrive at times 43 and 53).*

In the results of the optimization calculation we can further see that $x_{21} = 3$ minutes. Variable x_{21} models the time shift of the connections of line 2 in direction 1 relative to their earliest possible time positions. Line 2 in direction 1 acts as the outgoing line from transfer node 1, so the coordination period includes four connections serving transfer node 1 in the given direction, according to the input conditions of the task. The earliest possible service of transfer node 1 by line 2 connections running in direction 1 occur at times 00, 10, 20 and 30 (additional connections arrive at times 40 and 50). This means that after applying the time shift of $x_{21} = 3$ minutes, line 2 connections going in direction 1 at transfer node 1 will occur at times 03, 13, 23 and 33 (additional connections arrive at times 43 and 53).

In the results of the optimization calculation we can also see that $x_{22} = 3$ minutes. Variable x_{22} models the time shift of the connections of line 2 in direction 2 relative to their earliest possible time positions. Line 2 in direction 2 acts as the outgoing line from transfer node 1, so the coordination period includes four connections serving transfer node 1 in that direction, according to the input conditions of the task. The earliest possible line 2 services of transfer node 1 going in direction 2 occur again at times 00, 10, 20 and 30 (additional connections arrive at times 40 and 50). This means that after applying the time shift of $x_{22} = 3$ minutes, line 2 services of transfer node 1 going in direction 2 will occur at times 03, 13, 23 and 33 (additional connections arrive at times 43 and 53). As no alternating headway is applied on line 2 in direction 1, it is not necessary to carry out further calculations related to the identification of the time positions of the connections of the line in question, which were presented in the case of line 1.

In the results of the optimization calculation we can see that $x_{22} = 3$ minutes. Variable x_{22} models the time shift of the connections of line 2 in direction 2 relative to their earliest possible time positions. Line 2 in direction 2 acts as the outgoing line from transfer node 1, so the coordination period includes four connections serving transfer node 1 in that direction, according to the input conditions of the task. The earliest possible line 2 services of transfer node 1 going in direction 2 occur again at times 00, 10, 20 and 30 (additional connections arrive at times 40 and 50). This means that after applying a time shift of $x_{22} = 3$ minutes, line 2 services of transfer node 1 going in *direction 2 will occur again at times 03, 13, 23 and 33 (additional connections arrive at times 43 and 53). As no alternating headway is applied on line 2 in direction 2, it is not necessary to carry out further calculations related to the identification of the time positions of the connections of the line in question, which were presented in the case of line 1.*

In the results of the optimization calculation we can further see that $x_{32} = 9$ *minutes. Variable* x_{32} *models the time shift of the connections of line 3 in direction 2 relative to their earliest possible time positions. Line 3 in direction 2 acts as an incoming line to transfer node 2, so the coordination period includes three connections serving transfer node 2 in the given direction, according to the input conditions of the task. The earliest possible services of transfer node 2 by line 3 going in direction 2 occur at 00, 10 and 20 (additional connections arrive at 30, 40 and 50). This means that after applying the time shift of* $x_{32} = 9$ *minutes, the service of transfer node 2 by line 3 going in direction 2 will occur at times 09, 19 and 29 (additional connections arrive at times 39, 49 and 59). As no alternating headway is applied on line 3 in direction 2, it is not necessary to carry out further calculations related to the identification of the time positions of the connections of the line in question, which were presented in the case of line 1.*

In the example, 3 coordination links were defined: in transfer node 1 from line 1 connections going in direction 1 to line 2 connections going in direction 1 and to line 2 connections going in direction 2 and at transfer node 2 from line 3 connections going in direction 2 to line 1 connections going in direction 2. According to the output of the results, the transferring passengers should incur a total time loss of 35 minutes, which we check in the following via the values of the variables h_{uilkjs}*, or their sum.*

Connections arriving on line 1 going in direction 1 serve transfer node 1 at times 03, 11, 18 and 26. The transfer time between the connections of the incoming line 1 going in direction 1 and the connections of the outgoing line 2 going in direction 1 is 5 minutes. Passengers arriving on line 1 connections going in direction 1 at times 03, 11, 18 and 26 will therefore be ready for departures of line 2 connections going in direction 1 at times 08, 16, 23 and 31. Line 2 connections going in direction 1 depart at 03, 13, 23 and 33. The time loss of transferring passengers will therefore be in the order of 5 *minutes (08 → 13), 7 minutes (16 → 23), 0 minutes (23 → 23) and 2 minutes (31 → 33). And since in the case of transfers we consider a model value of the volume of transferring passengers from each service of an arriving line to be 1, then the passengers transferring from the connections of line 1 going in direction 1 to the connections of line 2 going in direction 1 will generate a time loss of 14 person-minutes.*

The same situation arises in the case of a transfer from connections of line 1 going in direction 1 to connections of line 2 going in direction 2. Connections arriving on line 1 going in direction 1 serve transfer node 1 at times 03, 11, 18 and 26. The transfer time between the connections of the incoming line 1 going in direction 1 and the connections of the outgoing line 2 going in direction 2 is 5 minutes. Passengers arriving on line 1 connections going in direction 1 at times 03, 11, 18 and 26 will therefore be ready for departures of line 2 connections going in direction 2 again at times 08, 16, 23 and 31. Line 2 connections going in direction 2 depart at 03, 13, 23 and 33. The time loss of transferring passengers will therefore again be in the order of 5 minutes (08 → 13), 7 minutes (16 → 23), 0 minutes (23 → 23) and 2 minutes (31 → 33). And since in the case of transfers we consider a model value of the volume of transferring passengers from each connection of the arriving line to be 1, then the passengers transferring from the connections of line 1 going in direction 1 to the connections of line 2 going in direction 2 will again generate a time loss of 14 person-minutes.

Next, let us check the value of the time loss of transferring passengers for the third coordination link. Arriving line 3 connections going in direction 2 serve transfer node 2 at 09, 19 and 29. The transfer time between incoming line 3 connections going in direction 2 and the outgoing line 1 connections going in direction 2 is 6 minutes. Therefore, passengers arriving on connections of line 3 going in direction 2 at 09, 19 and 29 will be ready for the departures of line 1 connections going in direction 2 at 15, 25 and 35. Line 1 connections going in direction 2 depart at 05, 12, 20, 27 and 35. The time loss of transferring passengers will therefore again be in the order of 5 minutes (15 → 20), 2 minutes (25 → 27) and 0 minutes (35 → 35). And since in the case of transfers we consider a model value of the volume of transferring passengers from each connection of the arriving line to be 1, then passengers transferring from *connections of line 3 going in direction 2 to connections of line 1 going in direction 2 will generate a time loss of 7 person-minutes.*

Since the values of the time loss of transferring passengers in all three coordination links are equal to 14 person-minutes, 14 person-minutes and 7 person-minutes, the total time loss is equal to 35 person-minutes.

At the end of the interpretation of the example results, the values of the variables $z_{uilkjsp}$ *will be interpreted. From the point of view of the results of the optimization calculation, the values of variable 1 listed above are important, as they represent the formation of the transfer links. For the results of Example 7.3* $z_{1111212} = 1$, $z_{1112213} = 1$, $z_{1113213} = 1$, $z_{1114214} = 1$, $z_{1111222} = 1$, $z_{1112223} = 1, z_{1113223} = 1$, $z_{1114224} = 1$, $z_{2321123} = 1$, $z_{2322124} = 1$ *and* $z_{2323125} = 1$ *are valid. The values of the variables are intended to confirm the interpretation of the results made in the previous paragraphs.*

Variable $z_{1111212}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection (fourth index) and line 2 (fifth index), direction 1 (sixth index) and its connection 2 (seventh index). Variable* $z_{1112213}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 2 (fourth index) and line 2 (fifth index), direction 1 (sixth index) and its connection 3 (seventh index). Variable* $z_{1113213}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 3 (fourth index) and line 2 (fifth index), direction 1 (sixth index) and its connection 3 (seventh index). Variable* $z_{1114214}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 4 (fourth index) and line 2 (fifth index), direction 1 (sixth index) and its connection 4 (seventh index).*

Variable $z_{1111222}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 1 (fourth index) and line 2 (fifth index), direction 2 (sixth index) and its connection 2 (seventh index). Variable* $z_{1112223}$ *and its value 1 represents the formation* *of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 2 (fourth index) and line 2 (fifth index), direction 2 (sixth index) and its connection 3 (seventh index). Variable* $z_{1113223}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 3 (fourth index) and line 2 (fifth index), direction 2 (sixth index) and its connection 3 (seventh index). Variable* $z_{1114224}$ *and its value 1 represents the formation of a coordination link at transfer node 1 (first index) between line 1 (second index), direction 1 (third index) and its connection 4 (fourth index) and line 2 (fifth index), direction 2 (sixth index) and its connection 4 (seventh index).*

Variable $z_{2321123}$ *and its value 1 represents the formation of a coordination link at transfer node 2 (first index) between line 3 (second index), direction 2 (third index) and its connection 1 (fourth index) and line 1 (fifth index), direction 2 (sixth index) and its connection 3 (seventh index). Variable* $z_{2322124}$ *and its value 1 represents the formation of a coordination link at transfer node 2 (first index) between line 3 (second index), direction 2 (third index) and its connection 2 (fourth index) and line 1 (fifth index), direction 2 (sixth index) and its connection 4 (seventh index). Variable* $z_{2323125}$ *and its value 1 represents the formation of a coordination link at transfer node 2 (first index) between line 3 (second index), direction 2 (third index) and its connection 3 (fourth index) and line 1 (fifth index), direction 2 (sixth index) and its connection 5 (seventh index).*

8 ECONOMIC ASPECTS OF THE USE OF THE RESULTS OF THE METHODOLOGY

The results of the project reflected in the methodology bring a number of economic benefits for its users.

The main economic benefit is related to the time savings for users of public transport services (passengers) that they will incur in their transit process based on the use of the public transport system in which the results of the methodology have been applied. However, direct quantification of the economic benefits to passengers is problematic. In particular, the fact that the same time saved has a different financial value for each passenger using public transport poses problems in the determination of its actual value.

In terms of societal benefits, the application of the results of the methodology can contribute to increasing the number of passengers carried on public transport vehicles. This can also have the positive effect of reducing the volume of passenger car traffic, especially in urban areas. A decrease in the volume of individual car traffic will lead to a decrease in the amount of congestion and therefore also to a reduction in the so-called external costs of transport (costs related to, for example, elimination of the consequences of road accidents, reduction of the negative effects of individual car traffic on the environment in the inner city, etc.).

The increase in the volume of passengers in public transport vehicles also increases their use and brings more funds to the budgets of transport operators (e.g. as a result of passengers repeatedly purchasing long-term tickets, etc.), which also increases the financial stability of transport operators' revenues. Increasing the stability of revenues of public transport operators can consequently contribute to better predictability of their economic management and therefore, for example, to the possibility of more effective investment in the modernisation of the vehicle fleet, which will bring additional economic effects to the budget of public transport operators.

Ultimately, the results of the methodology may also have a positive impact on providers of subsidies to public transport, in the form of lower amounts of funds required to cover demonstrable losses of transport operators resulting from the difference in costs of transport operators and fare revenues from passengers.

From the point of view of transport operators that would consider acquiring a timetable designer decision support system based on the methodology, the economic evaluation will need to include the costs related to the acquisition of the optimization software and the costs related to the training of timetable designers to work with the software.

9 CONCLUSIONS

The methodology deals with the issue of network node time coordination of public transport connections and aims to contribute to increasing the attractiveness and therefore the competitiveness of public transport in relation to passenger car transport. It deals with the operating conditions on public transport networks in which headway operation is applied. It concerns cases where either a headway of the same value always elapses between connections in the timetable on individual coordinated lines (constant headway) or permitted combinations of headways of different lengths occurring in a prescribed order (alternating headway).

Mixed integer linear programming was chosen to solve the research problem, so the proposed approach is an optimization approach. The basis of the proposed optimization approach is a mathematical model with an optimization criterion - the total time loss of all transferring passengers at all transfer nodes arising within the so-called coordination links. Each coordination link (a requirement to ensure the continuity of the connections of coordinated lines) is always characterized by an ordered seven within the solved problem, where the data contained therein identify the transfer node in which the coordination link is to be implemented, the identification data of the line, direction, possibly also the exact connections from which passengers are to transfer, the identification data of the line and direction to which the coordination link is to be formed from the given connections and the average volume of transferring passengers.

An essential part of the considerations leading to the creation of the mathematical model are calculations related to the determination of the length of the coordination period and the identification of the number of connections included in the coordination task in question. The calculation procedure leading to the required results is given in subsection 6.4.2. The following subsections are directly devoted to the process of creating mathematical models for different categories of operational situations occurring on public transport networks with headway operation. Specifically, these are the following operational variants:

1. operational variant with the same constant headways between connections on the arrival and departure of lines to/from all transfer nodes,

2. operational variant with different headways between connections on the arrival and departure of lines to/from transfer nodes,
3. operational variant with a constant headway between the connections on the arrival to the transfer node and an alternating headway between the connections on the departure from the transfer node,
4. operational variant with alternating headway between the connections on the arrival to the transfer node and constant headway between the connections on the departure from the transfer node,
5. operational variant with alternating headway between the connections on the arrival to the transfer node and alternating headway between the connections on the departure from the transfer node.

The actual network model will then be created either by direct application of one of the above variants or by combinations of the models of the individual variants presented in subsections 6.6 - 6.10.

Chapter 7 then demonstrates practical examples of solving and building models using specific illustrative examples of operational situations in which network node time coordination is required.

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LIST OF FIGURES

LIST OF TABLES