Příloha č. 10
ke zprávě o průběhu přijímacího řízení na vysokých školách pro akademický rok 2018 – 2019
na ČVUT v Praze Fakultě dopravní

Informace o písemných přijímacích zkouškách

(úplné zadání zkušebních otázek či příkladů, které jsou součástí přijímací zkoušky nebo její části, a u otázek s výběrem odpovědi správné řešení)

Doktorské studijní programy - Matematika

(část „Matematická analýza")

Varianta 1

Vypočtěte integrál $\displaystyle\int_0^1 \frac{(4x+9)\,\mathrm{d}x}{2x^2-5x-3}$.

ŘEŠENÍ: Integrovanou funkci rozložíme na parciální zlomky

$$\frac{4x+9}{2x^2-5x-3} = \frac{3}{x-3} - \frac{2}{2x+1}\,.$$

Pak dostaneme

$$\int_0^1 \frac{(4x+9)\,\mathrm{d}x}{2x^2-5x-3} = \int_0^1 \left(\frac{3}{x-3} - \frac{2}{2x+1}\right)\mathrm{d}x = \Big[3\ln|x-3| - \ln|2x+1|\Big]_0^1 = 3\ln 2 - 4\ln 3\,.$$

Najdětete řešení Cauchyovy úlohy

$$x_1' = 3x_1 - 2x_2\,, \qquad x_2' = 4x_1 - x_2\,, \qquad\qquad x_1(0) = 2\,, \quad x_2(0) = 1\,.$$

ŘEŠENÍ: Jedná se o homogenní soustavu dvou lineárních diferenciálních rovnic prvního řádu $\mathbf{x}' = \mathbf{A}\mathbf{x}$, kde $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$. Její charakteristická rovnice

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = \lambda^2 - 2\lambda + 5 = 0\,,$$

má komplexně sdružená řešení $\lambda_{1,2} = 1 \pm 2\mathrm{i}$. Stačí najít vlastní vektor matice $\mathbf{A}$, který odpovídá jednomu z vlastních čísel. Například pro

$$\lambda = 1 + 2\mathrm{i} \;\longrightarrow\; \mathbf{v} = \begin{pmatrix} 1 \\ 1-\mathrm{i} \end{pmatrix}.$$

Když sestrojíme odpovídající komplexní řešení $\mathbf{z} = \mathrm{e}^{\lambda t}\mathbf{v}$, jsou dvě lineárně nezávislá řešení soustavy $\mathbf{x}_1 = \mathrm{Re}\, z$ a $\mathbf{x}_2 = \mathrm{Im}\, z$. V naem případě je

$$\mathbf{z} = \mathrm{e}^t(\cos 2t + \mathrm{i}\sin 2t)\begin{pmatrix} 1 \\ 1-\mathrm{i} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^t\cos 2t \\ \mathrm{e}^t(\cos 2t + \sin 2t) \end{pmatrix} + \mathrm{i}\begin{pmatrix} \mathrm{e}^t\sin 2t \\ \mathrm{e}^t(\sin 2t - \cos 2t) \end{pmatrix}.$$

Tedy soustava má dvě lineárně nezávislá řešení

$$\mathbf{x}_1 = \begin{pmatrix} \mathrm{e}^t \cos 2t \\ \mathrm{e}^t(\cos 2t + \sin 2t) \end{pmatrix}, \qquad \mathbf{x}_2 = \begin{pmatrix} \mathrm{e}^t \sin 2t \\ \mathrm{e}^t(\sin 2t - \cos 2t) \end{pmatrix}$$

a její obecné řešení je

$$\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2\,, \quad \text{tj.} \quad \begin{aligned} x_1 &= C_1\mathrm{e}^t \cos 2t + C_2\mathrm{e}^t \sin 2t\,, \\ x_2 &= C_1\mathrm{e}^t(\cos 2t + \sin 2t) + C_2\mathrm{e}^t(\sin 2t - \cos 2t)\,, \end{aligned}$$

kde C_1 a C_2 jsou konstanty. Pro ty získáme z počátečních podmínek rovnice

$$C_1 = 2\,, \quad C_1 - C_2 = 1\,, \quad \text{tj.} \quad C_1 = 2\,, \quad C_2 = 1\,.$$

Řešení Cauchyovy úlohy tedy je

$$x_1(t) = \mathrm{e}^t(2\cos 2t + \sin 2t)\,, \qquad x_2(t) = \mathrm{e}^t(\cos 2t + 3\sin 2t)\,.$$

Pro jaký poloměr r a výšku v bude mít uzavřená válcová nádoba s objemem $V = 16\pi\,\mathrm{m}^3$ nejmenší povrch?

ŘEŠENÍ: Povrch válce je $S = 2\pi r^2 + 2\pi r v$ a jeho objem $V = \pi r^2 v = 16\pi$. Tedy

$$v = \frac{16}{r^2} \quad \text{a} \quad S(r) = 2\pi\left(r^2 + \frac{16}{r}\right).$$

Funkce $S(r)$ může nabývat miimum v bodech, kde

$$S'(r) = 4\pi\left(r - \frac{8}{r^2}\right) = 0\,, \quad \text{tj.} \quad r = 2\,.$$

Protože $\lim\limits_{r\to 0_+} S(r) = \lim\limits_{r\to+\infty} S(r) = +\infty$, je bodě $r = 2$ skutečně minimum.
Rozměry hledaného válce tedy jsou $r = 2\,\mathrm{m}$ a $v = 4\,\mathrm{m}$.

Varianta 2

Vypočtěte integrál $\displaystyle\int_1^2 \frac{(2x-7)\,\mathrm{d}x}{4x^2+5x-6}$.

ŘEŠENÍ: Integrovanou funkci rozložíme na parciální zlomky

$$\frac{2x-7}{4x^2+5x-6} = \frac{1}{x+2} - \frac{2}{4x-3}\,.$$

Pak dostaneme

$$\begin{aligned} \int_1^2 \frac{(2x-7)\,\mathrm{d}x}{4x^2+5x-6} &= \int_1^2 \left(\frac{1}{x+2} - \frac{2}{4x-3}\right)\mathrm{d}x = \\ &= \Big[\ln|x+2| - \tfrac{1}{2}\ln|4x-3|\Big]_1^2 = \ln 4 - \ln 3 - \tfrac{1}{2}\ln 5\,. \end{aligned}$$

Najdětete řešení Cauchyovy úlohy

$$x'' + 2x' + 10x = 0\,, \qquad x(0) = -1\,, \quad x'(0) = 4\,.$$

ŘEŠENÍ: Jedná se o homogenní lineární diferenciální rovnici druhého řádu a konstantními koeficienty. Její charakteristická rovnice $\lambda^2 + 2\lambda + 10 = 0$ má řešení $\lambda_{1,2} = -1 \pm 3\mathrm{i}$. Proto jsou dvě lineárně nezávislá řešení rovnice $x_1 = \mathrm{e}^{-t}\cos 3t$ a $x_2 = \mathrm{e}^{-t}\sin 3t$ a její obecné řešení

$$x(t) = C_1\mathrm{e}^{-t}\cos 3t + C_2\mathrm{e}^{-t}\sin 3t\,,$$

kde C_1 a C_2 jsou konstanty. Z počátečních podmínek pro ně dostaneme rovnice

$$C_1 = -1\,, \quad -C_1 + 3C_2 = 4\,, \quad \text{tj.} \quad C_1 = -1\,, \quad C_2 = 1\,.$$

Tedy řešení Cauchyovy úlohy je

$$x(t) = \mathrm{e}^{-t}\big(\sin 3t - \cos 3t\big)\,.$$

Najděte nejdelší sečnu elipsy $x^2 + 3y^2 = 12$, která prochází bodem $[0; -2]$.

ŘEŠENÍ: Nechť je $[x; y] \neq [0; -2]$ bod elipsy, tj. platí $x^2 + 3y^3 = 12$. Vzdálenost tohoto bodu ob vrcholu $[0; -2]$ je $d = \sqrt{x^2 + (y+2)^2}$. Když dosadíme za x^2 dostaneme úlohu najít maximum funkce

$$d(y) = \sqrt{16 + 4y - 2y^2} \quad \text{na intervalu} \quad -2 \le y \le 2\,.$$

Tato funkce může mít extrem v bodě, kde

$$d'(y) = \frac{2 - 2y}{\sqrt{16 + 4y - 2y^2}} = 0\,, \quad \text{tj.} \quad y = 1\,.$$

Protože $d(-2) = 0$, $d(2) = 4$ a $d(1) = 3\sqrt{2} > 4$, nastává maximum pro $y = 1$.
Hledané body tedy jsou $[3; 1]$ nebo $[-3; 1]$.

Příloha č. 11
ke zprávě o průběhu přijímacího řízení na vysokých školách pro akademický rok 2018 – 2019
na ČVUT v Praze Fakultě dopravní

Informace o písemných přijímacích zkouškách

(úplné zadání zkušebních otázek či příkladů, které jsou součástí přijímací zkoušky nebo její části, a u otázek s výběrem odpovědi správné řešení)

Doktorské studijní programy - Matematika

(část „Pravděpodobnost a matematika")

Test A

jméno a příjmení

1. **Příklad:**
   Určete aritmetický průměr dat, zadaných tabulkou hodnot x_i a četností n_i

x_i	1	2	3
n_i	15	60	25

$[\overline{x} = 2.1]$

Postup řešení:
$P = (1 * 15 + 2 * 60 + 3 * 25)/(15 + 60 + 25)$.

2. **Příklad:**
   Náhodná veličina má obor hodnot 1, 2 a 3. Platí $P(1) = 2p$, $P(2) = 0.5 - p$ a $P(3) = p$. Nakreslete pravděpodobnostní a distribuční funkci této náhodné veličiny.

$[p = 1/4$, hus. $(1/2,\ 1/4,\ 1/4)$, dist. $(0,\ 1/2,\ 3/4,\ 1)]$

Postup řešení:
Součet hodnot pravděpodobnostní funkce je jedna.

3. **Příklad:**
   Metodou momentů sestrojte statistiku pro odhad parametru p alternativního rozdělení s pravděpodobnostní funkcí

$$f(x) = p^x(1-p)^{1-x} \qquad \text{pro} \quad x \in \{0;1\}.$$

$[\hat{p} = \overline{x}]$

Postup řešení:
Porovnat první moment souboru (spočtený podle definice z hustoty) a výběru (výběrový průměr).

Test B

jméno a příjmení

1. **Příklad:**
   Určete medián z dat, zadaných tabulkou hodnot x_i a četností n_i

x_i	15	20	30
n_i	15	32	47

$[\tilde{x} = 25]$

   Postup řešení:
   $15 + 32 + 47 = 94, \quad 94/2 = 47.$ Prostředek je mezi 20 a 30.

2. **Příklad:**
   Náhodná veličina má hustotu pravděpodobnosti

$$f(x) = 2^{-k} \sin(x) \qquad \text{pro} \quad x \in (0, \pi).$$

   Určete hodnotu reálné konstanty k.

$[k = 1]$

   Postup řešení:
   Integrál z pravděpodobnostní funkce je jedna.

3. **Příklad:**
   Dokažte nestrannost výběrového průměru jako statistiky pro odhad střední hodnoty souboru.

$$[E[\tfrac{1}{n}\sum X_i] = \tfrac{1}{N}\sum E[X_i] = \mu]$$

   Postup řešení:
   Viz výsledek.

DATE: June 13 2018

NAME:

TOTAL: /15 pts

PART 1 (7.5 pts, 0.5 each sentence)

Choose the answer which best fits the sentence:

1. He was always interested in trying new things so it was _____ that he should write a book.
   A. claimed **B. suggested** C. told D. known

2. After he _____ his knee during a match, he had to stop playing professionally.
   A. had injured B. has injured C. was injuring D. was injured

3. The history of _____ is a fairly short one.
   A. aeroplane **B. the aeroplane** C. an aeroplane D. the aeroplanes

4. They are _____ fascinating people that everyone would like to know them!
   A. so **B. such** C. such a D. so much

5. In 1992, Ranulph Fiennes and Mike Stroud _____ to walk across Antarctica.
   A. set out B. got away C. set up D. got by

6. I wish I _____ the file that you sent me.
   A. hadn't lost B. didn't lose C. wouldn't have lost D. wouldn't lose

7. You'd rather he stayed in England, _____?
   A. hadn't you B. shouldn't you C. didn't you **D. wouldn't you**

8. He is the teacher who joined the staff in 1988 and _____here ever since.
   A. works B. have worked **C. has been working** D. is working

9. His inability _____ decisions is one of his many weaknesses.
   A. to make B. to do C. to take D. to create

10. He is very well known _____ the world.
    A. all in **B. all over** C. in all D. at

11. I am hoping to apply _____ a job with an international company.
    A. on B. to C. with **D. for**

12. The job offers a number of _____ such as free travel and medical care.
    A. prospects B. qualifications **C. benefits** D. experiences

13. Learning a second language is not the same _____ learning a first language.
    A. than **B. as** C. like D. to

14. I stopped _____ to Rose when I bumped into her on the street.
    A. talk B. talking **C. to talk** D. to talking

15. The website was _____ by a husband and wife team.
    A. made out B. put on **C. set up** D. got off

PART 2 (7.5 pts, 0.5 each question)

You are going to read an article about what some highly respected scientists think could happen in the future. For questions 1-15, choose from the scientists A-E. The scientists may be chosen more than once. When more than one answer is required, these may be given in any order.

Where is science going?

We asked five well-known scientists to give us their views on some of the most exciting developments we could see in the next few years.

A

Susan Greenfield, brain scientist

We are going to see a way of linking information technology with brain research. Already scientists have managed to place an electrode in the brain of a paralysed man who was unable to move any part of his body. This allowed him to use electrical activity of his brain to move the cursor on a computer screen. This sort of research could have enormous value for people with brain damage and it is important that we continue to look for ways to help patients with conditions like this. However, one thing that worries me is that scientists don't use their imaginations enough. So, I'm involved in a project in which young scientists are sent out into the developing world to help people in rural areas. I hope they'll come back more aware of other cultures, and more imaginative and creative about the way they do science.

B

Sir David Attenborough, wildlife broadcaster

Natural history film-making has always been very much affected by technology. With the latest technical advances, we can take a close-up film of something as minute as the eye of a fly. This means we'll be able to make incredible films about insects like spiders and scorpions which actually have very dramatic lives. And I think these documentaries will change people's opinions of nature programmes and show them that they are not just dry, educational stuff. If you can get people to watch cartoon programmes like *The Simpsons* on TV, you ought to be able to get them to watch films about scorpions too. As far as the future in general is concerned, I think the biggest change on the way is that there will be too many people around. There will be less and less of emptiness, the wilderness, left in the world.

C

Sir Patrick Moore, astronomer

The most important thing in the field of space research will be to find out whether there is life on Mars. If we knew that we were not the only intelligent beings, then it would have enormous implications for our future and how we look at it - our whole philosophy in fact. Personally, I think that there is life on other planets. Why should we be unique? But whether or not we'll find this out all depends on money, and if we go on wasting our money on wars instead of using it for this type of research then of course we won't get anywhere. I also think that in the future we should spend far more than we do now on medical research, especially when the number of people in the world is rising at such a rapid rate. After all, one of the worst dangers we face is disease and we need to understand how to cure it.

D

Lord Robert Winston, fertility doctor

I am excited by the idea of extending women's reproductive life. I don't want to see women of 60 having babies. But women are now healthier, fitter and more able to have babies at 45 or 50 than they've ever been. However, there is a lot of public suspicion of developments like this, and many people question whether we

should be doing this sort of work. I think a lot of the problem is ignorance, and what we should be doing is making science in schools relevant so that students don't think of it as a boring subject. And I think there should be recognition that things like television programmes where you have 5 million viewers – I've never had less than 5 million viewers for any programme – are very influential as well.

E

Trevor Baylis, inventor of the wind-up radio

I believe that we'll have to go back to go forwards. We have to stop depending on electricity for everything and look instead at other sources of power. For example, the wind-up radio I invented just works by clockwork, without needing batteries or anything, and we now have torches and lights powered like this as well. In general, I think greed is one of the biggest problems that we have to face up to. It's wrong that some people make billions and billions of pounds when there are other people in the world who can't even afford to buy a packet of cornflakes. We have to do something about it.

Which scientist mentions that (write down the relevant letter)**:**

modern scientific equipment can help us see very small things?	1 __B_
the distribution of wealth among the world's population is unfair?	2 __E_
scientists need to get experience of different ways of living?	3 __A_
the mass media can help people to learn about science?	4 _B__, 5 _D__
some living creatures experience very exciting events?	6 _B__
the increase in the world's population could be a problem?	7 __B_, 8 _C__
people may be unsure of the value of some scientific advances?	9 _D__
investment in scientific research should be regarded as a priority?	10 _C__
children need to be taught science in an interesting way?	11 _D__
scientists need to find better ways of dealing with illness?	12 _A__, 13 __C_
research could affect people's view of what the future holds?	14 __C_
progress may not need advanced technology?	15 __E_